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Mc COLL'S
ALGEBRAICAL EXERCISES AND PROBLEMS
WITH
ELLIPTICAL SOLUTIONS.

ALGEBRAICAL EXERCISES

AND

PROBLEMS:

WITH ELLIPTICAL SOLUTIONS:

ALSO AN

Appendix on Factors.

BY

HUGH McCOLL, B.A.

LONDON UNIVERSITY

NEW EDITION.

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PREFACE.

This little book is intended for beginners in Algebra; but a single glance at its contents will make it evident that it is not meant to supersede any of the ordinary systematic treatises on this subject.

The main object of the work is to train beginners in the application of the elementary operations of algebra to the solution of problems. The elliptical solutions at the end of the exercises have been written with this view. The pupil is supposed to copy these on paper, filling up the blanks as he goes along. Each link in a chain of reasoning is thus brought before him in logical succession; but he is expected to join the links and complete the chain himself. If he does all this successfully and without the assistance of his teacher, the latter may feel satisfied that his pupil has understood both the problem proposed and its solution; while, on the other hand, if the pupil meets with any difficulty, the teacher can at once see where the difficulty arises, and remove it. This secures economy of labour on the part both of teacher and pupil. The latter, instead of wasting time and energy in attempting to solve by his own unaided powers those problems which, upon a fair trial, he finds too difficult for him, is enabled to concentrate his attention successively upon single points, and so conquer his difficulties in detail; while the former is spared the trouble of explaining difficulties which his pupil might not perhaps find very formidable if left to himself, and may reserve his assistance for those cases which really require it.

Many of the examples are so framed as to make hardly any demand upon the reasoning powers, while at the same time they afford good practice in the four elementary rules, the simplification of fractions and other expressions, and in the mechanical operations of algebra generally. The pupil will find the answers to these at the end of the exercises, but, generally speaking, no solutions.

Though a large portion of the examples are original, a greater number still are either copied verbatim from questions set at various competitive examinations, or adapted from these so as to suit the peculiar nature of this little work. As an instance of such adaptation, suppose the question in the examination-paper to be:

'Simplify the expression $\frac{ax}{a-x}-\frac{2ax^2}{a^2-x^2}$.'

This may be amplified into:

'A person's present age is $\frac{ax}{a-x}$ years; how old was he $\frac{2ax^2}{a^2-x^2}$ years ago?'

or

'Two adjacent sides of a rectangle are respectively x feet and $\frac{a}{a-x} - \frac{2ax}{a^2-x^2}$ feet? what is the area?'

or

'A tradesman bought x articles at $\frac{2ax}{a^2-x^2}$ pence each, and sold them at $\frac{a}{a-x}$ pence each, and he gained by the transaction: how much?'

or

'A railway train travelling from A to B, a distance of $\frac{ax}{a-x}$ miles, stops on its way at the end of x hours at a station, C. What is the distance between B and C, supposing the train to travel at a uniform rate of $\frac{2ax}{a^2-x^2}$ miles an hour?'

The answer to the first question is of course $\frac{ax}{a+x}$ years; to the second $\frac{ax}{a+x}$ square feet; and so on.

The pupil will often find it an instructive exercise to verify the result obtained by giving any numerical value he pleases to each letter.

At the end of the exercises will be found an Appendix, in which are given a few simple methods *on one uniform plan,* for resolving expressions into their elementary factors. The general character of these methods may be seen from the summary at the end. The diagonal order of multiplication, it will be observed, is no essential part of the general principle, and the teacher may adopt it or not at his discretion. One advantage of these methods is that the pupil can, in every case, see, by mere inspection, what method it will be most advantageous for him to employ, and consequently loses no time in deliberation.

Any remarks or suggestions from teachers or learners will be thankfully received by the author,

HUGH McCOLL.

CLASSIFICATION OF PROBLEMS, &c.

AVERAGES.—xvi, 5; xviii, 4; xlviii, 4.

DIGITS, DECIMALS, ETC.—xlix and l.

EQUATIONS OF IDENTITY.—iii, 9; xlvii, 2.

GAMES.—xix, 1, 2; xxv, all.

LENGTHS AND AREAS.—vii, 4; x, 1, 2; xvi, 4; xx, 2; xxi, 2, 3; xxii, 4; xxiii, 4; xxiv, all; xl, 5, 6; and nearly all in xli, xlii, xliii, xliv, and xlv.

MEANING OF THE *Negative Sign*, AND OF EXPRESSIONS WHICH TAKE THE FORM $\frac{0}{0}$ ETC.: viii, 6; xxviii, 1, 2, 3; xlvii, 3, 4.

MIXTURES OF TEAS, LIQUIDS, ETC.—xv, 7; xxxviii, all; xxxix, all; xlviii, 2.

PERCENTAGES.—xxxvi, all; xlviii, 2; l, 5.

PROBLEMS ON *Mistakes* AND *Misprints*.—xi, 7; xii, 4; xiv, 3; xxvii, 4.

RULES AND EXAMPLES ON *Changes of Signs*.—xlvi.

ROWING WITH AND AGAINST A CURRENT.—xxix, all; xxx, all.

SELLING BY THE *Yard* AND BY THE *Square Yard*.—xxiv, 5.

TIME, RATE, AND DISTANCE.—ii, 6; v, all; vi, 6; viii, 3; xii, 2; xiv, 2; xvi, 3; xvii, all; xxii, 6; xxxv, 5; and others.

WORK.—xv, 4, 5; xix, 4; xxi, 1, 6; xxii, 1; xxvi, 3; xxxi, 4, 5; xxxii, 7; xxxiv, 4, 5; xxxvii, all.

The rest are miscellaneous examples not easy to classify.

EXERCISES.

EXERCISE I.

1. If x denote a certain number or fraction, what is the algebraical expression for 3 times this number or fraction? for half the number? for one-fourth of it? for two-thirds of it? for the number which exceeds it by 3? for the number which falls short of it by 3? for the square of it? for the square root of it?

2. If a certain number (or fraction) be denoted by the algebraical expression $5x+y-6$, what is the algebraical expression for 5 times the number? for the half of it? for three-fourths of it? for the number which exceeds it by 4? for the number which exceeds it by 6? for the number (or fraction) which falls short of it by 6? for the square of it? for the square root of it?

3. One number is denoted by x, and another by $x-2$. Express in algebraical language the statement that half the first number is equal to two-thirds of the other. Assuming this statement to be correct, find the numbers.

4. A certain number is represented by x, and another by $2x+8$; and five-sixths of the first number exceeds one-fourth of the other by 2. Find the numbers represented by x and by $2x+8$.

EXERCISE II.

1. A person bought an article for $3x^2-4x-6$ pence, and sold it for x^2+5x+3 pence; how much did he gain? Verify the result on the supposition that $x=5$.

2. A person bought an article for $3a+b-c$ pounds, and sold it for $2a+2b-c$ pounds; how much did he lose?

3. What will $3x^2-2x+4$ articles cost at $3x$ shillings each? Verify the result (1) when $x=3$; (2) when $x=4$.

4. Find the cost of $2x^2-5x-4$ articles at x^2-2 pence each.

5. If $x-2$ articles cost $4x^3-11x^2+4x+4$ pence, what is the price per article?

6. If a man walk at the rate of x miles an hour, how far will he walk in 6 hours? How far does he walk in 40 minutes, at the same rate?

EXERCISE III.

1. One person contributed $3x^2-5x+6$ pounds, another contributed $13x-6x^2-3$ pounds, and a third contributed $3x^2-7$ pounds; how much did they all contribute?

2. Three persons, A, B, and C, contributed altogether a sum of 20 pounds, of which A contributed $2x^2-3x+2$ pounds, B contributed $3x+3-x^2$ pounds, and C contributed $6x-x^2-3$ pounds. Find the numerical value of x; and thence find the numerical value of each contribution.

3. Three persons contributed a sum of money; the first contributed $x^3-3ax^2+3a^2x-a^3$ pounds, the second contributed $3ax^2-6a^2x+3a^3$ pounds more than the first, and the third contributed 3 times as much as the first and second together. What was the whole sum contributed?

4. What was the whole sum contributed by the 3 persons spoken of in the preceding example, if $x=3$, and $a=1$?

5. Three benevolent gentlemen, A, B, and C, raised among them a sum of 309*l.* to set up a poor widow in business; B contributed 3*l.* less than A, and C contributed twice as much as A and B together. How much did each contribute?

6. x denotes a certain fraction, and $\frac{2x}{3}$ denotes another; and the sum of the fractions is $\frac{1}{7}$. Find the fractions.

7. A certain number is denoted by x; another and a smaller number is denoted by $\frac{x-2}{3}$; and the difference between the numbers is 6. Find the numbers.

8. A boy had an algebraical problem to solve which required him to find the yearly incomes of two persons, A and B. Assuming that A's yearly income was x pounds, he reasoned correctly from some of the conditions of the problem that B's yearly income must be $10x-\frac{x-\frac{x}{4}}{5}$ pounds; and from others he reasoned again correctly that B's yearly income must be $5(2x-3)$ pounds. Find the two incomes?

9. Can the numerical value of x be found from the equation $\frac{x-3}{2}-\frac{x-4}{5}=\frac{3x-7}{10}$? Give a reason for your answer.

EXERCISE IV.

1. How many articles could be bought for $12x^4+20x^3y-19x^2y^2+15xy^3-21y^4$ pence, if each article cost $3x^2+5xy-7y^2$ pence?

2. A certain person had x^4 shillings in his purse; how much of this sum had he left after paying for $x^2-2ax+3a^2$ articles, which cost $x^2+2ax-3a^2$ shillings each?

3. A certain person's present age is $6-5x+x^2$ years; when will he be x^2+5x+6 years old?

4. A company of x persons bought a ship's cargo for $(x^2-2x)(x^2+3x+1)$ pounds, and afterwards sold the whole for $2x^4$ pounds; what was each person's share of the profits? Verify this (1) when $x=10$; (2) when $x=9$.

5. A man's age 5 years ago was $\frac{5}{7}$ of what his age will be 5 years hence; find his present age.

6. A tradesman bought a certain quantity of goods for x pounds, and sold the whole for ax pounds, making a total profit of a^2-1 pounds. Find x; that is to say, express x in terms of a.

EXERCISE V.

1. How long will it take a person to travel a distance of $a^2x^2-b^2y^2+2bcy-c^2$ miles, at the rate of $ax-by+c$ miles an hour?

2. A certain man walked a distance of $a^2x^2+acx-2c^2$ miles in $2c+ax$ hours; at what rate did he walk per hour?

3. A person has a journey of a^4-x^4 miles to accomplish; for the first a^2-x^2 hours he walks at the rate of $2ax$ miles per hour; at what rate must he walk so as to accomplish the remainder of the journey in $(a-x)^2$ hours?

4. A man engages to perform at a uniform rate a certain distance on horseback in 10 hours; but when he is halfway he increases his speed by 2 miles an hour, and thus arrives at his journey's end one hour and a quarter before the time stipulated. Required, the distance he performed, and the rate at which he started.

EXERCISE VI.

1. A man is now x years old, and in $\frac{x}{2}$ years he will be 30. Find x.

2. $\frac{x-3}{5}$ denotes a certain number, $\frac{x-2}{3}$ denotes a larger number, and the difference between the numbers is $\frac{x+40}{12}$. Find the numbers.

3. The number represented by x exceeds the number represented by $\frac{x-2}{3}$ by $7\frac{1}{3}$. Find the numbers.

4. Two fractions are denoted by $\frac{x}{2}$ and $\frac{3}{4}(2-x)$ respectively; and two others by x and $2x+1\frac{3}{8}$ respectively; and the sum of the first two is equal to the difference between the other two. Find the fractions.

5. A sum of 30*l.* was divided equally among x men, and a sum of 45*l.* among $2(x-2)$ women; and a man's share

was equal to a woman's. Find how many men and women there were.

6. A man walked a distance of 13 miles in $3\frac{3}{4}$ hours; during the first 3 hours he walked at the rate of $x-\frac{1}{2}$ miles an hour, and during the remaining three-quarters of an hour he walked at the rate of $x-\frac{2}{3}$ miles an hour. Find x, and the rates at which he walked.

EXERCISE VII.

1. A man bought a horse for x pounds, and sold him for bx pounds; and he gained b^2-1 pounds by the transaction. Find x.

2. In paying for a bill which amounted to x shillings, the change out of a five-pound note was $\frac{3}{4}(4x+20)-\frac{4x-25}{5}$ shillings. Find x.

3. Two persons, A and B, each bought an article for x shillings. A, by selling his for m shillings, gained just half as much as B lost by selling his for n shillings. Required, A's gain and B's loss, the answers to be expressed in terms of m and n.

4. A line x inches long is divided into two parts; the length of one part is $\frac{x-3}{2}$ inches, and the length of the other is $\frac{3}{4}(x-6)$ inches. Find the length of the line, and the lengths of the parts.

5. If a certain number, denoted by x^2-1, be divided by another number denoted by $x+1$, the quotient will be 24. Find the numbers.

EXERCISE VIII.

1. A tradesman bought a certain quantity of goods for $\frac{5x}{2}-\{\frac{3}{2}x-(x-\frac{2}{3})\}$ pounds, and sold the same for $2x$ pounds; and he gained by the transaction: how much? Verify the result (1) when $x=20$; (2) when $x=100$; (3) when $x=4$.

2. If I sell for $2x$ pounds an article which cost me $\frac{x^2-a^2}{x-a}$ pounds, how much shall I gain? Verify the result (1) when $x=10$, and $a=6$; (2) when $x=120$, and $a=20$. When $x=8$ and $a=10$, the answer is -2 pounds; what does the negative sign indicate?

3. A person walked from a certain town, A, to another town, B, in $2a$ hours. During the first $a+x$ hours he walked at a uniform rate of a^2-ax+x^2 miles an hour, and for the rest of the time at a uniform rate of a^2+ax+x^2 miles an hour. What is the distance between the towns?

4. One side of a certain rectangle is $a^2+b^2+c^2-ab-bc-ca$ feet long, and the side at right angles to this side is $a+b+c$ feet long; what is the area of the rectangle?

5. The area of a certain rectangle is $x^4+x^2y^2+y^4$ square feet, and the breadth of the rectangle is x^2-xy+y^2 feet; what is the length?

6. A boy had once an algebraical problem to solve which required him to find the distance of a certain person from a certain town, A. He began by assuming that the person was x miles to the north of A; he then reasoned correctly from some of the conditions of the problem that he must be $5x-\frac{x-17}{2}$ miles to the south of another town, B; and, taking other conditions of the problem into account, he reasoned again correctly that he must be $\frac{7-\frac{x}{2}}{3}+1\frac{1}{2}$ miles to the south of B. Finish the solution of the problem. Find also the distance between A and B.

EXERCISE IX.

1. A person bought $2x$ articles at $\frac{x}{2}-3$ shillings each, and $8x-\frac{4}{3}x$ articles at $\frac{3x}{4}$ shillings each; what did all the articles cost him?

2. A person bought $\frac{x}{3}-2$ articles at 3 pence each, and $\frac{x}{4}-1$ articles at 4 pence each, and the whole cost him 1*s.* 2*d.* How many articles did he buy at 4 pence each?

3. A gentleman gave a bank-note to pay for a bill which amounted to $\frac{2a}{2x-a}$ pounds; what change was due to him, the value of the note being $\frac{(2x+a)^2}{4x^2-a^2}$ pounds?

4. A man bought $\frac{x-5}{3}$ articles at a shilling each, and $x-4$ articles at 9 pence each; he gave a sovereign to pay for the whole, and received 3 shillings in change. Find the numerical value of x; and thence find how many articles he bought altogether.

5. If $5-2x$ articles cost $2x^3-5x^2-10x+25$ pence, what is the price per article?

Verify the result (1) when $x=1$, and (2) when $x=2$.

6. Find the greatest common measure and the least common multiple of $10x^3-6x^2-4x$ and $5x^3-2x^2-5x+2$.

EXERCISE X.

1. What is the area of a square, each side of which is $\frac{x^2}{2}-\frac{3}{4}x-1$ inches?

2. The area of a certain rectangle is $1+\frac{5}{6}x+\frac{1}{2}x^2+\frac{10}{3}x^3-x^4$ square feet, and the length of one side is $x+\frac{2}{3}-\frac{x^2}{3}$ feet; what is the length of the adjacent side?

3. In what time will a person walk a distance of $a^2+b^2-c^2-2ab$ miles at the rate of $b-a-c$ miles an hour?

4. If $a+\frac{ax}{a-x}$ men can do a certain work in $a-\frac{ax}{a+x}$ days, in what time could one man do it at the same rate?

5. What is the least common multiple of $2x$, $4x^2-1$, $(2x+1)^2$, $6(2x-1)^2$?

EXERCISE XI.

1. By what must $2a^2+7a+3$ be multiplied so that the product shall be $2a^3-7a^2-46a-21$.

2. Find the G.C.M. of $21x^3-15x^2-14x+10$ and $42x^3+54x^2-28x-36$.

3. What must be added to $\frac{1}{a+b}$ so that the sum shall be $\frac{2a}{a^2-b^2}$?

4. From what must $\frac{x}{x^2-(a+b)x+ab}$ be subtracted so that the remainder shall be $\frac{x}{x^2-(a-b)x-ab}$?

5. What must be subtracted from $\frac{x^2+y^2}{x^2-y^2}$ so that the remainder shall be $\frac{x-y}{x+y}$?

6. By what must $4x^4-2x^3-12x^2+3x+9$ be divided so that the quotient shall be $2x^2-3$?

7. From indistinctness in the printing of a division sign in an examination paper in arithmetic, the persons examined could not tell whether they were required to work a sum in division of fractions or in subtraction of fractions. Some worked the sum correctly on the one supposition, and some on the other, and, by a curious coincidence, they all got the same result, which was $4\frac{1}{2}$. Find the sum proposed.

EXERCISE XII.

1. If a certain algebraical expression be divided by $1-\frac{1}{1+x}$ the quotient will be $1-x^2$. Find the expression.

2. A person has a journey of $2a^3-ax^2-x^3$ miles to perform; how much of the journey will remain to be performed when he has travelled a^2+ax+x^2 hours at the rate of $a-x$ miles an hour?

3. A person bought a certain quantity of tea for $\frac{a}{a+b}$ pounds, and sold it for $\frac{a}{a-b}$ pounds; how much did he gain by the transaction? He divided his profits equally among $2ab$ persons; how much did each receive?

4. A boy, on being asked what $\frac{3}{4}$ of a certain fraction was, divided the fraction by $\frac{3}{4}$, and thus got an answer which exceeded the true answer by $\frac{35}{72}$. What was the true answer? What was the boy's answer?

EXERCISE XIII.

1. A man's purse contained x sovereigns, $\frac{3x}{10}$ shillings, and $\frac{2x}{5}$ sixpences; and the whole sum was 10*l.* 5*s.* Find the numerical values of x, $\frac{3x}{10}$, and $\frac{2x}{5}$.

2. In a certain money-box there were x sovereigns, $\frac{5x+1}{2}$ half-sovereigns, $\frac{2}{3}(7x-6)$ half-crowns, $11-x$ florins, $5x$ shillings, and $6x$ sixpences, and the whole sum amounted to 10*l.* 5*s.* Find the number of half-crowns.

3. The money in a man's purse consisted of sovereigns, shillings, and half-crowns; there were twice as many half-crowns as there were sovereigns, and the number of shillings was three-fourths of the number of half-crowns. The whole sum amounted to 2*l.* 13*s.* How many were there of each coin?

4. Two shepherds, A and B, owning a flock of sheep, agree to divide it. A takes 144 sheep, and B takes 184 sheep, paying 70*l.* to A. Required the value of a sheep.

EXERCISE XIV.

1. A merchant bought a certain quantity of tea for $\frac{m}{(a+b)^2}$ pounds, and sold it for $\frac{m}{(a-b)^2}$ pounds, and he gained $4x$ pounds by the transaction. Find x.

2. A travels at the rate of $\frac{a^3}{a^2+b^2}$ miles an hour, and B travels at the rate of $\frac{a^3}{a^2-b^2}$ miles an hour; how far will B be in advance of A in a^4-b^4 hours, supposing that they start from the same place at the same time and in the same direction?

3. A boy once, in working a sum in division of decimals, made a mistake in his pointing; he placed the decimal point in the quotient one place further to the right than he ought to have done, and thus got an answer which exceeded the true answer by $2\frac{1}{4}$. What was the true answer?

4. The area of a certain square is $\frac{x^2}{4}-3x+9$ square feet; what is the length of one side?

EXERCISE XV.

1. A man's present age is $2x-\{b-(x-2b)\}$ years; when will he be $x-\{2b-(3a+2b-x)\}$ years?

2. A company of $x-y$ persons engaged to pay in equal shares a bill which amounted to $x^3-3xy-y^3-1$ pounds, but one of them failed to do so; how much had each of the others to pay in consequence?

3. How many foreign coins, each worth x^2+3y^2-2xy pence, should be received in payment for a bill which amounts to $\frac{1}{12}x^4+\frac{3}{4}y^4+\frac{1}{6}x^2y^2$ shillings?

4. If $a+x$ men can perform a certain piece of work in $a^2+2ax+x^2$ days, in what time could one man do it, assuming that all the men work equally well, and at a uniform rate?

5. When $\frac{1}{1+x}$ of a work has been performed, how much will still remain to be done? Verify the result (1) when $x=1$; (2) when $x=2$; (3) when $x=3$.

6. A tradesman sold a certain quantity of goods for $\frac{x}{x-a}$

pounds, thus making a profit of $\frac{x}{a+x}$ pounds. How much did the goods cost him?

7. How much tea at $\frac{a}{12}$ shillings per lb., must be mixed with m lbs. at b pence per lb., so that the mixture may be worth c pence per lb.?

EXERCISE XVI.

1. A man can do $\frac{a}{x}$ of a work in one hour; in what time can he do the whole?

2. If a man can perform $x+\frac{1}{1+\frac{x+1}{3-x}}$ of a work in one day, in what time can he do the whole?

3. A railway train travelled a distance of $x+y$ miles in $2a$ hours; for the first x miles it travelled at the rate of $a+b$ miles an hour, and for the next y miles it travelled at the rate of $a-b$ miles an hour. If it had travelled throughout at the rate of a^2+b^2 miles an hour, it would have accomplished the whole distance in $\frac{x-y}{2ab}$ hours. Find x and y.

4. The length of a certain rectangle is $x+1$ yards, the breadth is $3x-3$ feet, and the area is 15 square yards. Find x; and thence find the numerical length and breadth.

5. At a local examination the number of candidates at 24 places gave a certain average per centre; and this average is diminished by $2\frac{17}{24}$ if London and the 95 students which it furnishes be omitted. How many were there in all?

6. What is the least common multiple of x^2-a^2, $2x+2a$, $ax^2-2a^2x+a^3$?

EXERCISE XVII.

1. A ship has a voyage of $a^2x^4-(2ac+b^2)x^2+c^2$ miles to accomplish; in what time will it be accomplished, sup-

posing the ship to sail at a uniform rate of ax^2+bx-c miles an hour?

2. A man has a journey of $(a+b)(a-b)$ miles to make; for the first c hours he travels at the rate of $c-2b$ miles an hour. At what rate per hour must he travel so as to accomplish the remaining part of the distance in $a-b+c$ hours?

3. Two persons start at the same time from two towns, A and B, and walk in a straight line towards each other; the one walks at the rate of x miles an hour, and the other at the rate of $a-x$ miles an hour, and they meet in a^2+ax+x^2 hours. What is the distance between A and B?

4. Two persons start at the same time from two towns, A and B, and walk in a straight line towards each other; the one who starts from A walks at the rate of 4 miles an hour, and the other walks at the rate of $3\frac{1}{2}$ miles an hour. Supposing the distance between the towns to be 60 miles, when will they meet, and where?

EXERCISE XVIII.

1. A sum of $2x^3+4x^2-x-2$ pounds was divided equally among $2x^3-4x^2-x+2$ persons; how much did each person receive? Verify the result (1) when $x=3$; (2) when $x=4$.

2. A sum of $4x^2+15x-4$ shillings was divided equally among $4x^2-17x+4$ persons, and each person received 2 shillings. Find the numerical value of x, and verify the result.

3. A railway train goes over a distance of $(x+y)^2-z^2$ miles in $x+y+z$ minutes; at what rate does it travel per minute?

4. During the Lancashire distress, the first division of a public school contributed abx pounds, the second division contributed $x\{b(b+c)+a(a+c)\}$ pounds, and the third contributed $(b+c)(a+c)-a^2x$ pounds; what was the average sum contributed by each boy, there being $(a+b)x$ boys in the first division, a in the second, and c in the third?

EXERCISE XIX.

1. Two gamesters, A and B, each possessing a sum of $4a-2\{b+\frac{1}{2}(c-2d)\}$ pounds, played for a stake of $3[a+\{b-\frac{2}{3}(c-d)\}]$ pounds, which B won. How much had each after the game?

2. Two gamesters, A and B, each possessing a sum of $\frac{x}{a+b}$ pounds, played for a stake of $\frac{x^2}{a^2-b^2}$ pounds, which A won, and then A's money was 5 times as much as B's. Express x in terms of a and b.

3. How long will a sum of $\frac{x+a}{2(x-a)}-\frac{x-a}{2(x+a)}+\frac{2a^2}{x^2-a^2}$ pounds last if $\frac{2a}{x-a}$ shillings be spent every day?

4. Three persons, A, B, and C, executed a certain piece of work in 3 days. A alone worked the first day and performed a fraction of it, which we shall denote by x; B alone worked the next day and did $\frac{2}{5}$ of the remainder; after which $\frac{3}{10}$ of the whole work still remained, which C finished on the third day. How much of the work was done by A and B respectively?

5. Find the G.C.M. and the L.C.M. of $3a^4-6a^3b+5a^2b^2-6ab^3+2b^4$, and $2a^4-6a^3b+5a^2b^2-6ab^3+3b^4$.

EXERCISE XX.

1. Four arithmetical fractions are denoted respectively by the four algebraical expressions x, y, $\frac{3x-y}{3\frac{1}{3}}$, $\frac{x+\frac{4}{3}}{3}$; the sum of the first and second is 1, and the difference between the third and fourth (the fourth being the greater) is $\frac{1}{6}$. Find the fractions.

2. A straight line $2\frac{1}{4}$ feet long is divided into two parts; the length of one part is $\frac{10}{3}(x-y)$ inches, and the length of the other part is $\frac{17}{60}(x+y)$ feet. If the first part were diminished by 2 inches, and the other part by 1 inch, the

first part would be just half the length of the other. Find the numerical values of x and y.

3. A person bought an article for $\frac{9}{x}$ pounds, and sold it for $\frac{4}{y}$ pounds, and lost one pound by the transaction. If he had bought it for $\frac{18}{x}$ shillings, and sold it for 16 shillings, he would have gained $\frac{20}{y}$ shillings. Find x and y.

4. A privateer which runs at the rate of x miles an hour, seeing a ship 25 miles off, whose rate of sailing is $\frac{4x}{5}$ miles an hour, immediately gives chase, and overtakes her in $10\frac{1}{2}$ hours. Find x.

EXERCISE XXI.

1. A can perform a certain work in ax days, B can do the same in a^2-ax days, and C in a^2-x^2 days; how much of the work can A, B, and C, working together, do in one day?

2. The length of one side of a rectangle is $\frac{3}{2}x^2-x-\frac{1}{2}$ feet, and the length of the adjacent side is $x^2-\frac{3}{2}x+2$ feet; what is the area?

3. The area of a certain rectangle is x^3+x^2-4x+2 square feet, and the length of one side is x^2+3x-4 feet; what is the length of the adjacent side?

4. A man bought an article for $\frac{a}{x(a-x)}$ pounds, and sold it for $\frac{x}{a(a-x)}$ pounds; and he lost by the transaction: how much?

5. How many square feet are there in the area of a square whose side is $\frac{x}{2}-\frac{a}{2}-\frac{2}{b}$ yards?

6. A man and his wife consumed together a quantity of flour in 12 days, but when the man was away the same

quantity lasted the woman 30 days; in how many days would the man consume it alone?

EXERCISE XXII.

1. A can do a certain work in $(x-a)(a-b)$ days, and B can do the same work in $(b-x)(b-a)$ days; how much of the work could they do together in c days?

2. A man walked a distance of $4x^3-6x^2-6x+9$ miles, at the rate of $4x^2-9$ miles a day; in what time did he do it?

3. A company of $a+2b$ persons contributed $a+2b$ shillings each. Of this sum $c-3d$ persons received each a sum of $c-3d$ shillings. The rest was divided equally among $a+2b+c-3d$ persons. How much did each of these last receive?

4. A rectangle is $5a-\{2x-(a+x)+(3a-2x)\}$ feet long and $3\{a-5(a-x)+4(\frac{5}{4}a-\frac{4x}{3})\}$ feet broad. What is the area?

EXERCISE XXIII.

1. What is the L.C.M. of the four algebraical expressions $126a$, $14a^3(x^2-9)$, $7(x+3)^2$, $42a^2(x^3+27)$? What is the G.C.M. of the first two expressions? What is the G.C.M. of the last three?

2. Out of a total of x persons $\frac{2x}{3}$ persons received $6d.$ each, 4 received $9d.$ each, and the rest received $8d.$ each; express algebraically the whole sum received by the x persons in pence.

3. A sovereign was divided among a certain number of persons; two-thirds of them received $9d.$ each, one-ninth received $3s.$ each, two of them received $1s.$ each, and the rest received $1s.$ $6d.$ each; how many persons were there?

4. A rectangle 4 inches longer than it is broad is three-fourths of the size of another rectangle, which is 10 inches

longer than the first, but not so broad by 2 inches. Find the lengths and breadths of the two rectangles.

EXERCISE XXIV.

1. What length of paper will be required to cover a wall $a^3+4a^2x+4ax^2$ feet by $a^2-4ax+4x^2$ feet, the width of the paper being a^2-4x^2 feet?

2. A room is x^2 feet long, a^2-ax feet wide, and $a+x$ feet high; what is the area of the 4 walls, no allowance being made for doors or windows?

3. What would a rectangular field, $x^2+\frac{2}{3}ax-\frac{a^2}{2}$ yards long and $x^2-\frac{2}{3}ax-\frac{a^2}{2}$ yards broad, cost at 6 shillings per square yard?

4. How many squares, each side of which is $z-x-y$ inches, would be required to cover a rectangle $x+y+z$ feet long and $z-(x+y)$ feet broad?

5. By selling a piece of carpet $2\frac{1}{2}$ feet wide in proportion to its length at 6 shillings per *yard*, an upholsterer gained just 1*l.* more than he would have gained if he had sold the same piece at 6 shillings per *square yard.* Find the length of the piece.

EXERCISE XXV.

1. Two gamesters, A and B, each possessing $\frac{a^2+x^2}{a^2-x^2}$ shillings, played for a stake of $\frac{a^2-x^2}{a^2+x^2}$ shillings, which A won. How much had each after the game?

2. The two gamesters spoken of in the preceding example played a second game for a stake of $\frac{4a^2x^2}{a^4-x^4}$ shillings, which A lost; how much money had each after the second game?

3. Two persons, A and B, engaged at play, each with a sum of 10*l.* When they left off, A found that if B gave him

half-a-crown, he would then have just half as much money as B. How much did A lose?

4. Two persons, A and B, played for a stake of a pounds, which B won, and then for a stake of b pounds, which A won. At the end of the first game A's money was $\frac{3}{4}$ of B's; and at the end of the second game A's money was to B's as 5 to 4. How much had each when they began playing?

EXERCISE XXVI.

1. Find the G.C.M. and the L.C.M. of $4a^2x(20x^3-64x^2+57x-9)$ and $6ax^2(50x^3-95x^2+32x-3)$.

2. How many foreign coins, each worth $x-\frac{y}{2}$ pence, are equal in value to $\frac{2x}{y}-2+\frac{y}{2x}$ shillings?

3. If $x-\frac{2a}{3}$ men can perform a certain work in 4 days, in what time could $3ax-2a^2$ men do it, assuming that all the men work at the same uniform rate? If each of the second set of men performs as much work in 3 hours as each of the first set performs in 4 hours, in what time could the second set do it in this case?

4. A gentleman divided x^2-6 pence, all he had, equally among $x-2$ beggars; if he had had 2 pence more, each beggar would have received 8 pence. Find x; and thence find how much each beggar actually received.

EXERCISE XXVII.

1. A merchant bought a certain quantity of tea for $\frac{a^3-b^3}{a^2+ab+b^2}$ pounds, and sold the same for $a+b$ pounds; how much did he gain?

2. If an article be bought for $\frac{b(2a+b)}{a^2-b^2}$ shillings, and

sold for $\frac{ab}{a^2-b^2}-\frac{b}{b-a}$ shillings, what is the gain or loss?

3. By selling an article for $\frac{b}{a-\frac{b^2}{a}}$ pence, a person gained $\frac{\frac{ab}{b-a}}{a+b}$ pence; what did the article cost him?

4. From an oversight of the printer's, a simple equation in a certain mathematical text-book is given thus:

$$\frac{x}{a-b}-\frac{x}{\quad}=\frac{a^2}{a^2-b^2};$$

and the answer given at the end of the book is $\frac{a^2}{2b}$. Supply the missing denominator, and then verify your result by solving the equation.

EXERCISE XXVIII.

1. A person bought 100 articles for $\frac{10-x}{5}$ pounds, sold the lot for $\frac{x+7}{2}$ pounds, and gained on the whole $x+3$ pounds; how much did he gain on each article?

2. What is the numerical value of the expression $\frac{13a\left\{a^3-\left(\frac{x}{2}-\frac{a^2-6}{3}\right)\right\}(x-7)(a-x)^2}{a^2-3ax+x^2}$ where $a=25$ and $x=7$.

3. Show that, whatever value be given to x, the numerical value of the fraction $\frac{4(3x^2+5x-8}{8x^2-5x-3}$ will be the same as that of the fraction $\frac{4(3x+8)}{8x+3}$. Find the numerical value of each fraction, (1) when $x=3$; (2) when $x=2$; and (3) when $x=1$. Explain the peculiar form assumed by the first fraction in the last case.

4. Find the G.C.M. and the L.C.M. of the expressions $(2x-3)(2x^3-3x^2-2x+3)$ and $(2x-3)(2x^3-3x^2+2x-3)$.

EXERCISE XXIX.

1. A man can row at the rate of $\dfrac{a^3+\dfrac{1}{x^3}}{a^3-\dfrac{1}{x^3}}$ miles an hour in still water; at what rate can he row against a current which flows at the rate of $\dfrac{a^3-\dfrac{1}{x^3}}{a^3+\dfrac{1}{x^3}}$ miles an hour? At what rate will he travel if he rows *with* the current?

2. A man rowing with a current which flows at the rate of $\dfrac{x-y}{2x+2y}$ miles an hour, travels at the rate of $\dfrac{x+y}{x-y}$ miles an hour; at what rate would he travel if he rowed against the current?

3. If a boat's crew can row at the rate of a^2+x^2 miles an hour in still water, in what time could they row a distance of a^3-x^3 miles down a river which has a current of ax miles an hour? In what time could they row a distance of a^3+x^3 miles up the river?

4. A man rowed a distance of m miles down a river and back again in n hours. At what rate could he row in still water, supposing the river to have a current of one mile per hour?

EXERCISE XXX.

1. A boat's crew found that they could row down a river a distance of a miles in m hours, and the same distance up the river in n hours. Find the rate of the current in this part of the river.

2. A boat's crew can row down a river a distance of $\dfrac{4m^2n^2}{m^2-n^2}$ miles in m hours, and the same distance back in n hours. At what rate can they row in still water?

3. A and B are two villages on the banks of a river, separated by a distance of p miles. Two boatmen, each

exerting a force which in still water would carry him at the rate of m miles an hour, start at the same time from these two villages towards each other; where will they meet, supposing the river to have a current of n miles an hour, and A to be higher up the river than B?

EXERCISE XXXI.

1. Find an algebraical expression for the number of articles which at $-2x^2+2x+3$ pence each would cost $2x^4-12x^3+x^2+21x+9$ pence.

2. A man walked a distance of $-6x^4+4x^3-5x^2+6x+6$ miles in $2x^2+3$ hours; at what rate did he travel per hour?

3. A person bought $9x^2-16$ articles at $x+4$ pence each, and sold them at $4x+2$ pence each; he divided his profits equally among $4-3x$ persons; how much did each person receive?

4. A can do as much work in $ax-b$ hours as B can do in $ax+b$ hours; in what time could B accomplish a piece of work which A can do in $\frac{a^2x^2}{b^2}-1$ hours?

5. A can perform in $\frac{a^2x^2-b^2}{b^2}$ hours a piece of work which B can execute in $\left(\frac{ax+b}{b}\right)^2$ hours; how much of the work can they do together in one hour?

6. A certain fraction becomes equal to $\frac{2}{3}$ when 2 is added to its numerator; and it becomes equal to $\frac{1}{2}$ when 1 is subtracted from its denominator. Find the fraction.

EXERCISE XXXII.

1. Find the G.C.M. and the L.C.M. of the three expressions $a^2x^3(x-1)^2$, $ax^2(x^2-1)$, $a^3x(x^2+1)(x+1)^2$.

2. Prize money to the amount of $3x(a+2b)^2$ pounds was divided among $(b-c)^2$ officers and $a+b+c$ sailors. The

officers received $3x$ pounds each, and the sailors received $3y$ pounds each: express y in terms of x.

3. A sum of $\frac{9x^2}{4}-1$ shillings was divided equally among 21 persons, and each person received $3x+2$ pence. Find x, and verify your result.

4. A man buys a case of oranges at 6*d.* a dozen. He finds fifty spoiled, and, selling the remainder at the rate of 3 for 2*d.*, has 10*s.* 9*d.* for profit. How many oranges were there in the case?

5. If a^3-x^3 yards of cloth cost a^2+ax+x^2 shillings, what will a^2-x^2 yards cost at the same rate?

6. Two men on velocipedes start from a place A in the same direction, the one at a quarter-past 3, at the rate of 400 yards per minute; and the other at 20 minutes past 3, at the rate of 440 yards per minute. When will the second overtake the first, and at what distance from A?

7. A and B are two workmen; A alone takes 4 hours more to perform a certain work than B alone takes; also A and B together can perform the work in $3\frac{3}{4}$ hours. How long will each take to perform it alone?

EXERCISE XXXIII.

1. A man went out with x sovereigns in his purse; before he came back he bought $\frac{ax}{2}$ articles at $\frac{2}{x}-\frac{4}{3x}$ shillings each, and $2a$ articles at $\frac{15x}{2a}-\frac{x}{a}$ shillings each; how much money had he left after paying for all those articles?

2. An army which originally consisted of

$$8a-\left\{\frac{5}{2}a-x-\left[\frac{a}{2}-\left(\frac{3a}{2}-x\right)\right]\right\}$$

men lost $\frac{1}{3}$ of this number from war, sickness, and other causes; how many men were left?

3. A man's present age is $\frac{x}{a-b}$ years; what was his age

$\frac{x}{a+b}$ years ago? How old will he be in $\frac{x}{a+b}$ years? Verify this (1) when $x=10$, $a=1$, and $b=\frac{2}{3}$; (2) when $x=20$, $a=3$, $b=2$.

4. Two armies of $x-\frac{a}{a-b}$ men and $\frac{x}{2}-\frac{b}{a+b}$ men respectively fought a battle, in which the former lost $\frac{1}{4}$ of its number, and the other $\frac{1}{6}$ of its number. After the battle the number of men in the first army was to the number of men in the second as 9 to 10. Determine x; and thence find the original number of men in each army, a and b being understood to be known.

5. Twice the difference between two fractions is equal to $\frac{3}{4}$ of their sum; and the greater exceeds the less as much as half the less falls short of $\frac{3}{4}$. Find the fractions.

Exercise XXXIV.

1. How many books can be bought for

$$2ax^4-bx^3-\{2a(b+c)-c\}x^2+b(b+c)x-(bc+c^2)$$

pence at $2ax^2-bx+c$ pence each?

2. If $\frac{a^2-b^2}{(a-b)^3}$ cwts. of tea cost $\frac{a^2b+ab^2}{(a-b)^2}$ pounds, what is the price per cwt.?

3. A sum of $x^3-x^2-8x+12$ shillings was divided equally among $x+3$ persons, and each person received 4 shillings. Find numerically the number of persons.

4. $2x^3+9x^2-27$ labourers can do as much work in 3 hours as y labourers working at the same rate, can do in $4x^2-9$ hours. Express y in terms of x.

5. A and B performed a piece of work between them; $\frac{1-x}{1+x}$ denotes the fraction of it which A performed; how much of it did B perform? Verify the result (1) when $x=\frac{1}{2}$; and (2) when $x=\frac{2}{5}$.

6. The product of two fractions is equal to a times their sum and to b times their difference. Find the fractions.

EXERCISE XXXV.

1. How far will a train travel in $\frac{4}{9}x^2-\frac{2}{3}x+1$ hours at the rate of $1+\frac{2}{3}x+\frac{4}{9}x^2$ miles an hour?

2. In what time will a railway train travel a distance of $x^4-\frac{13ax^3}{6}+\frac{1}{2}a^2x^2+\frac{1}{3}a^3x$ yards at the rate of $\frac{2}{3}x^2-ax-\frac{1}{3}a^2$ yards per minute?

3. Two persons run a race, starting from the same place at the same time, the one at the rate of $3x^2+\frac{x}{2}-1$ yards per minute, and the other at the rate of $3x^2+2$ yards per minute; by how many yards will the former (supposing him to be the fastest runner) beat the latter in $\frac{x}{2}-3$ minutes?

4. A sum of a^2-x^2 shillings was divided equally among $\frac{a+x}{x}$ men; and a sum of $\frac{a^2-x^2}{a+x}$ shillings was divided equally among x women: by how much did a man's share exceed a woman's? Verify the result when $x=10$, $a=1$.

5. Two persons start from the same place at the same time and in opposite directions, the one at a uniform rate of m miles an hour, and the other at a uniform rate of n miles an hour; and in x hours they are separated by a distance of mn miles. Find x.

EXERCISE XXXVI.

1. If I buy any number of articles at the rate of m pence each, and sell them at the rate of $m+\frac{m}{4}$ pence each, how much do I gain per cent?

2. If I buy any number of articles at m pence a dozen, and sell them at $m-\frac{m}{3}$ pence a dozen, what is my loss per cent?

3. For what must I sell an article which cost me x shillings, so as to gain at the rate of 7 per cent?

4. A tradesman, in selling damaged goods, loses at the rate of 7 per cent; for how much does he sell an article which cost him x shillings? For how much will he sell an article which cost him $\frac{x}{8}-2$ pence?

5. I buy articles at the rate of a pence each, and sell them at the rate of b pence each; if I gain, at what rate per cent. do I gain; and if I lose, at what rate per cent. do I lose?

6. By selling tea at the rate of $\left(\frac{3x}{2}-3\right)$ shillings per lb., a tradesman gained at the rate of 17 per cent. on the prime cost? At what price per lb. did he buy it?

7. A tradesman bought his sugar at m£ per cwt., and sold it at x pence per lb.; what was his gain per cent.?

8. If goods be bought at a pounds per cwt. and sold at $\frac{x}{28}$ pence per lb.; what is the loss per cwt? What is the loss per cent.?

9. If a profit of 9 per cent. be made on the prime cost, at what price is anything sold which cost $3x-2y$ shillings?

10. Supposing a tradesman to lose at the rate of 9 per cent. on the prime cost, at what price does he sell anything which cost him $3x-2y$ shillings?

11. If anything be bought for x pence and sold for y pence, what is the gain per cent. on the prime cost? What would the gain per cent. be, if (contrary to the usual practice) it were reckoned *on the selling price?*

EXERCISE XXXVII.

1. What is the least common multiple of the four expressions a^2x^2, $4ax(x^2-a^2)$, $12(x^3+a^3)$, and $8(x^2-ax+a^2)$? What is the greatest common measure of the third and fourth?

2. When a man has performed $\frac{1}{1+\frac{1}{x}}$ of a work, how much of it will remain unfinished? Verify the result (1) when $x=1$; (2) when $x=2$; and (3) when $x=\frac{1}{2}$.

3. Three persons, A, B, and C, engaged to execute a work for $2x^2-2$ pounds; A did $\frac{1}{x-1}$ of the work, B did $\frac{1}{x^2-1}$ of it, and C did the remainder. They agreed to divide the money in proportion to the work done; how much should each receive?

4. A and B had to perform a certain work; A performed in 3 days a fraction of it which we will denote by x, and B did the rest in $3(1-x^2)$ days. Assuming that they worked uniformly, how much did each do in a day?

5. Two persons, A and B, can together perform a certain work in a days. They work together for b days, at the end of which A leaves off, and B alone completes the work in c days. How much of the work is thus done by each?

6. Two workmen contract to do a piece of work for 25*l.*, which they reckon will take them 75 days to execute. After working together for 45 days, one of them falls ill, and his mate completes the job in 50 days. How much is each entitled to receive?

7. If a man performs $\frac{a}{b}$ of a work in a day, in what time, at the same rate, can he perform the whole?

8. If a man can accomplish the whole of a work in $\frac{x}{a}$ days, how much of it can he do in one day, at the same rate?

9. If a man can do $\frac{(a-b)(b+c)}{ac}$ of a work in $b+c$ days, in what time could he do the whole?

EXERCISE XXXVIII.

1. A grocer mixed x lbs. of tea worth $x+y-z$ shillings per lb. with y lbs. worth $y+z-x$ shillings per lb., and z lbs. worth $z+x-y$ shillings per lb. What was the value of the mixture?

2. A grocer mixed x lbs. of tea which cost him $a+y$ shillings per lb. with y lbs. of tea which cost him $a-x$ shillings per lb., and he sold the mixture at b shillings per lb; how much did he gain on the whole? What does the result indicate when the numerical value of a is greater than that of b?

3. How many gallons of wine worth a^2-4 shillings per gallon must a wine-merchant mix with $a-2$ gallons worth $a+2$ shillings per gallon, so that the mixture may be worth a^2+4a+4 shillings?

4. A wine-merchant has two sorts of wine, one sort worth a shillings per quart, and the other worth b shillings per quart; from these he wants to make a mixture of n quarts worth c shillings per quart. How many quarts must he take from each sort?

5. Solve the equation:

$$\frac{5x}{2}\left(\frac{x}{2}-3\right)(2x-3)=0.$$

EXERCISE XXXIX.

1. If a gallons of wine be mixed with b gallons of water, what fractions of the whole will the wine and the water be respectively?

2. A wine merchant mixed $a+\frac{1}{2}$ gallons of water with $a^2-\frac{1}{4}$ gallons of wine; what fraction of the whole did the water constitute?

3. In a cask containing a mixture of wine and water, the ratio of the water to the wine is $2a-b : 2a+b$; how many quarts of water will there be in $4a^2-b^2$ quarts of this mixture? How many quarts of wine?

4. A cask, A, contains m gallons of wine and n gallons of water; and another cask, B, contains p gallons of wine and q gallons of water; how many gallons must be drawn from each cask so as to produce by their mixture b gallons of wine and c gallons of water?

5. What is the G.C.M. of $876(4x^2+4x-15)$, and $1825(10x^2-11x-6)$?

EXERCISE XL.

1. If $1-\frac{1-x^2}{1+x^2}$ ounces cost $1-\frac{1+x^2}{(1+x)^2}$ shillings, what is the price per ounce? What would $x+\frac{1}{x}$ ounces cost at the same rate?

2. If $4a^2-x^2$ men can do a certain work in $\frac{2a-x}{4a^2+4ax+x^2}$ days, in what time could 10 of them do it at the same rate?

3. What is the value of a square piece of ground, each side of which is $\frac{x}{2}-\frac{2}{3}$ yards, at 3*s.* per square yard?

4. In what time will a person walk a distance of $a^2-b^2-c^2+2bc$ miles, at the rate of $b-a-c$ miles an hour?

5. If the area of a rectangle be $a^2b^2\left(\frac{1+a}{1-a}-a\right)$ square feet, and the length of one side be $ab\left(a+\frac{a+a^2}{1-a}\right)$ feet, what is the length of the adjacent side?

6. The length of a rectangular piece of ground is to the breadth as 3 to 2; and at 9*d.* per square foot, the value of it would be 62*l.* 10*s.* Find the length and breadth.

EXERCISE XLI.

1. Find the G.C.M. and the L.C.M. of $30x^4-6x^3-50x^2+10x$ and $a(30x^4-6x^3+50x^2-10x)$.

2. The area of a certain rectangular field is $4a^2+b^2-c^2-4ab$ square yards, and the breadth is $b-2a-c$ yards. What is the length? What would the field be worth at $\frac{20abc}{b+c-2a}$ shillings per square yard?

3. One side of a rectangular field is $\sqrt{\left(\frac{4a^2}{9}-x^2\right)}$ yards long, and the adjacent side is $\sqrt{\left(\frac{4a^2}{9}+x^2\right)}$ yards long; what is the area?

4. The side of a square is $x+\frac{a}{2}$ yards, and the area of the square is $\frac{81}{4}a^2$ square feet. Express x in terms of a.

5. The length of a certain rectangle is $\sqrt{(x+\frac{2}{3})}$ feet; the breadth is $\sqrt{(x-\frac{2}{3})}$ feet; and the area is $\frac{2}{\sqrt{3}}$ of a square foot. Find the numerical value of x; and thence find the length and breadth of the rectangle.

6. The two adjacent sides of a rectangle are respectively x inches and $x+2a$ inches; and the area is $8a^2$ square inches. Find x and $x+2a$.

EXERCISE XLII.

1. The side of a certain square is $\sqrt{x}-\sqrt{a}$ inches. What is the area of the square?

2. The two adjacent sides of a rectangle are, respectively $\sqrt{x}+\sqrt{a}$ inches and $\sqrt{x}-\sqrt{a}$ inches. What is the area of the rectangle?

3. The length of a certain rectangle is $\sqrt{(x+2)}$ feet, and the breadth is $\sqrt{(x-2)}$ feet. What is the area?

4. A rectangle $\sqrt{(x+2a)}$ yards long and $\sqrt{(x-a)}$ yards broad, has the same area as a square whose side is $\sqrt{x}$ yards. Find x; and thence find the length and breadth of the rectangle.

5. By how much does the square, whose side is $3\sqrt{(x^2-a^2)}$ feet, exceed in area the square whose side is $3x-\sqrt{a}$ feet?

6. The square whose side is $3\sqrt{(x^2-a^2)}$ feet exceeds the square whose side is $3x-\sqrt{a}$ feet by $a(5-9a)$ square feet. Find x.

7. The area of a rectangle is $\sqrt{(a+b)^2-c^2}$ square inches, and the length is $\sqrt{(a+b+c)}$ inches. What is the breadth?

EXERCISE XLIII.

1. The area of a certain square is x^2+a^2 square inches; what is the length of one side?

2. The area of a certain square is x^2-6x+9 square feet; what is the length of one side?

3. The area of one square is $9x-5$ square yards; the area of another square is $9x+7$ square yards; and the side of the first is 6 feet shorter than the side of the other. Find x; and thence find the lengths of the sides of the squares.

4. A certain square is equal in area to a certain rectangle; and the side of the square exceeds the breadth of the rectangle by 6 feet, but falls short of the length of the rectangle by 15 feet. Required, the length and breadth of the rectangle.

5. The rectangle, whose length is x feet and whose breadth is $x-a$ feet, is equal in area to the square whose side is $\sqrt{(x^2-2a^2)}$ feet. Find x; and thence find the area of the rectangle or square.

EXERCISE XLIV.

1. The area of a certain square is $4x^2-12x+9$; what is the algebraical expression for the length of one side, it being understood that the unit of area is the square described upon the unit of length?

2. One side of a rectangle is $4ax$ feet, and the adjacent side is $3ax$ yards; what is the algebraical expression for the length of one side of a square equal in area to this rectangle, a yard being taken as the unit?

3. One side of a certain right-angled triangle is 14 inches

shorter than the hypothenuse, and the other 7 inches shorter than the hypothenuse. Find the length of each side, and the area of the triangle.

4. A man bought $\sqrt{(ax)}$ articles at $\frac{\sqrt{a}-\sqrt{x}}{\sqrt{a}+\sqrt{x}}$ shillings each, and sold them at $\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$ shillings each, and he gained by the transaction; how much? Verify the result (1) when $a=25$ and $x=1$; and (2) when $a=25$ and $x=9$.

5. Find the G.C.M. and the L.C.M. of $6x^4-6x^3-6x-6$ and $4x^4-4x^3+8x^2-4x+4$.

Exercise XLV.

1. The area of a certain square is $a^2+b^2+c^2-2(ab+bc-ca)$ square inches; what is the length of its side? Verify the result (1) when $a=3$, $b=2$, and $c=1$; (2) when $a=3$, $b=2$, and $c=0$; (3) when $a=1$, $b=1$, $c=1$.

2. $\sqrt{x}+1$ articles cost $\sqrt{x}-1$ pence each; and the whole cost 8*d*. Find x.

3. The area of a certain rectangle is $x-3$ square yards, and one side is $3\sqrt{x}+\sqrt{27}$ feet; what is the length of the adjacent side?

4. The length of one side of a rectangle is $\sqrt{x}+a$ feet, the area is $x-a^2$ square feet, and the length of the side adjacent to the first is $2a$ feet. Find x.

5. A rectangle 4 inches longer than it is wide is equal to another rectangle 10 inches longer than the first, but only half as wide. Find the length of each rectangle.

6. Which is greater, the fraction $\frac{abc}{a+b+c}$ or the fraction $\frac{abc}{a+b-c}$, and how much? The quantities a, b, c are understood to be positive.

Exercise XLVI.

Rule 1.—The sign of any algebraical expression is reversed (that is to say, the numerical value of the expression changes from positive to negative, or from negative to positive) if we change the sign of every term. For example, the expressions $a^2-2ax-b(a+b)$ and $2ax+b(a+b)-a^2$ are numerically equal, whatever numbers the letters a, b, and x may represent, *but they are of opposite signs.* If the numerical value of the former be positive, that of the latter is negative; and *vice versâ.*

Rule 2.—An expression remains unaltered in value if we change the sign of every term, *and also the expression sign.* The following are three examples:

(1) $a-x = -(x-a)$.
(2) $+(2ax-a^2-x^2) = -(a^2-2ax+x^2)$.
(3) $-\{a^2-2ax-b(a+b)\} = +\{2ax-a^2+b(a+b)\}$.

Rule 3.—Any product or fraction remains unaltered in value if we change the signs of an *odd* number of factors *and also the product or fraction sign.* The odd factors changed in sign may be all multiplying factors, or all dividing factors; or some may be multiplying factors and the rest dividing factors. The following three examples will sufficiently illustrate the rule:

(1) $+a(b-a) = -a(a-b)$.

By changing $b-a$ into $a-b$, we change the sign of *one* factor (see Rule 1); and by changing the + at the beginning of the expression into −, we change the product sign.

(2) $-(x-a)(b-a)^2(c-b) = +(x-a)(a-b)^2(b-c)$.

Here, by changing $(b-a)^2$ into $(a-b)^2$ we change the signs of *two* factors, for $(b-a)^2$ is only an abbreviation for $(b-a)(b-a)$, and $(a-b)^2$ is an abbreviation for $(a-b)(a-b)$; and by changing $c-b$ into $b-c$ we change the sign of another factor (see Rule 1). Thus we change the signs of *three* factors altogether; and consequently, in order that the expression may remain unaltered in value, we change the product sign from − to +.

(3) $+\dfrac{(x-a)(c-a)(b-a)}{(c-x)^3(x-b)^2} = -\dfrac{(x-a)(a-c)(a-b)}{(x-c)^3(b-x)^2}$.

Here, by changing $c-a$ into $a-c$, and $b-a$ into $a-b$, we change the signs of *two* multiplying or numerator factors, and by changing $(c-x)^3$ into $(x-c)^3$ and $(x-b)^2$ into $(b-x)^2$ we change the signs of *five* (see the remarks on the preceding example) dividing or denominator factors. We thus change altogether the signs of *seven* factors, an odd number; and we consequently change the fraction sign from + to −.

Rule 4.—Any product or fraction remains unaltered in value if we change the signs of an *even* number of factors, *and leave unchanged the product or fraction sign.* The following example will sufficiently illustrate the rule:

$$+\frac{(a^2-b^2)(b+a)(b-a)^2}{(d-c)(x-a)^3} = +\frac{(a^2-b^2)(a+b)(a-b)^2}{(c-d)(a-x)^3}.$$

Here, by changing $(b-a)^2$ into $(a-b)^2$ we change the signs of *two* equal numerator factors (see the remarks on Rule 3); and by changing $(x-a)^3$ into $(a-x)^3$, and $(d-c)$ into $c-d$, we change the signs of *four* denominator factors, three of which are equal. We thus change altogether the signs of *six* factors, and must not therefore change the fraction sign, since *six* is an even number.

It is hardly necessary to state that the sign of the factor $b+a$ is not changed by changing it into $a+b$, though beginners sometimes, from thoughtlessness, make this mistake.

After these changes of signs it is easy to see that the numerator of the above fraction might be written

$$(a-b)^3(a+b)^2.$$

Simplify the following expressions:

(1) $+\dfrac{(x-a)^2(a-x)^3(a^2-x^2)}{(x^2-a^2)(x^3-a^3)}.$

(2) $\dfrac{a}{a-b}-\dfrac{b}{b-a}.$

(3) $\dfrac{x-a}{a^2-b^2}-\dfrac{x-a}{(b-a)^2}.$

(4) $\dfrac{1}{a+b}+\dfrac{a+b+b^2}{(b+a)(a^2-b^2)}+\dfrac{1}{b^2-a^2}$.

(5) $\dfrac{\dfrac{a^2-b^2}{b-a}}{\dfrac{b^3-a^3}{a-b}}+\dfrac{b^2-a^2}{a^2+ab+b^2}$.

(6) $\dfrac{x(a-x)^2(x-a)}{(x^2-a^2)(a^2-x^2)}-\dfrac{x(x-a)}{(a+x)^2}$.

(7) $(x-a)(a-x)$.

EXERCISE XLVII.

1. What values of x satisfy the equation

$$5x(6x^2-37x+6)(2x^2+x-1)=0\,?$$

2. What values of x satisfy the equation

$$\frac{3x^2-5}{3}-5=\frac{3x^2-20}{3}\,?$$

3. Whatever value we give to x, the algebraical expression $\dfrac{x^2-4}{x-2}$ is equal to the expression $x+2$. Verify this (1) when $x=3$; (2) when $x=4$; (3) when $x=100$. In what sense is this true when $x=2$?

4. A person bought a certain quantity of goods for $\dfrac{96}{x^2+x-6}$ pounds, and sold the same quantity for $\dfrac{48}{x^2-x-12}$ pounds; how much did he gain? Verify the result when $x=5$. What is the numerical value of the result when $x=8$? What does the negative value indicate?

5. A man, starting from a place A, walked $\dfrac{6}{x-3}$ miles eastward; he then retraced his steps and walked $\dfrac{8}{x-2}$ miles westward, and then stopped. How far was he to the east of A when he stopped? Verify the result when $x=4$. What does the result indicate when $x=8$?

EXERCISE XLVIII.

1. Two parties, A and B, trade together, the share of A being twice that of B. Having agreed to dissolve partnership, A takes one-quarter of the stock-in-trade, B takes the remainder and pays A 500*l*. What is the stock-in-trade worth?

2. Find how much water must be mixed with 40 gallons of spirit, which cost 15 shillings a gallon, so that by selling the mixture at 12 shillings a gallon there may be a gain of 10 per cent. on the outlay.

3. A plays at chess with B, winning 3 games out of 4; and afterwards with C, winning 2 out of 3; at the end of 21 games he has won 15. How many did he play with each?

4. In an examination of 40 candidates, one of them gained only 120 marks, and he lowers the average of the marks of the 40 by 20 marks below the average of the marks of the other 39. What was the average of the marks of the candidates?

EXERCISE XLIX.

1. What number is denoted in the ordinary notation by the digits x, y, z, when x is the first digit to the left, and y the next? What number will be represented by the same digits when the above order is reversed?

2. What number is represented in the ordinary decimal notation by the digits x, y, m, n, when arranged in the order in which we have written them? What number will be denoted by the same digits if x and m exchange positions, and also y and n?

3. What number will be denoted by the digits m, n, x, y, z, in the order in which we have written them, if a decimal point be placed between the digit n and the digit x?

4. What number would be denoted by the digits x, y, z,

m, in the order in which we have written them, if 8 instead of 10 were the base of our scale of notation?

5. A certain number is expressed by two digits, and the digit in the tens' place exceeds by 3 the digit in the units' place. What is the algebraical expression for the number, the digit in the units' place being denoted by x?

6. A certain number consists of two digits; the digit in the tens' place is denoted by x, and the product of the digits is 24. What is the algebraical expression for the number?

7. Two numbers in the ordinary notation each consist of two digits; reverse the digits in one number, and you get the other number; the ratio of the numbers is 3 to 8, and the sum of the digits is 9. Find the numbers.

8. Two digits to the right of a decimal point express a certain decimal fraction; subtract ·18 from the fraction, and the digits in it will exchange places; multiply this result by $\frac{7}{4}$, and the digits will again exchange places and give the original decimal fraction. Find this fraction.

EXERCISE L.

1. A certain repeating decimal fraction is expressed by one digit x to the right of the decimal point; what is the algebraical expression for the fraction?

2. A certain decimal fraction is expressed by three digits to the right of the decimal point, all three repeaters; x denotes the first digit after the point, y the next, and z the next. What is the algebraical expression for the fraction?

3. x denotes the first of two digits to the right of a decimal point, and y the next; what is the algebraical expression for the fraction if the digit y alone repeats?

4. A certain decimal fraction is expressed by two digits to the right of the decimal point, the second digit being a repeater. Add $\cdot 0\dot{8}$ to the fraction, and the digits will exchange places, the second digit to the right of the decimal point being, as before, a repeater. Add $\cdot 08\dot{6}$ to the original

fraction, and the digits will, as before, exchange places, but none of them will repeat. Find the fraction.

5. The purchaser of an estate lets it at once at a yearly rent which is 3 per cent. of the purchase-money. He leaves part of the purchase-money unpaid, and the interest on this part at 4 per cent. per annum absorbs $\frac{2}{3}$ of the rent. After a time he is able to pay off 2,000*l.* of his debt, and thenceforward, after paying the interest on the remainder, he has 180*l.* yearly left out of his rent. What was the purchase-money of the estate?

ANSWERS AND ELLIPTICAL SOLUTIONS.

I.

3. 8, 6.
4. 12, 32.

II.

1. $9+9x-2x^2$ pence.
2. $a-b$ pounds.
3. $9x^3-6x^2+12x$ shillings.
4. $2x^4-5x^3-8x^2+10x+8$ pence.
5. $4x^2-3x-2$ pence.
6. $6x$ miles ; $\frac{2}{3}x$ miles.

III.

1. $8x-4$ pounds.
2. They contributed respectively 11*l.*, 3*l.*, 6*l.*
3. $8x^3-12ax^2+4a^3$ pounds.
4. 112*l.*
5. A contributed x pounds, suppose ; then B must have contributed .... pounds, and C must have contributed ..... pounds. The whole sum contributed by the three must consequently have been .... pounds. But we are told that the whole sum contributed by the three was 309 pounds. We have therefore the equation

$$\ldots\ldots = 309,$$

from which we get $x = 53$.

The three contributions were therefore 53*l.*, 50*l.*, 206*l.*

6. $\frac{3}{35}$, $\frac{2}{35}$.
7. 8, 2.
8. 100*l.*, 985*l.*
9. No ; for it is an equation of identity, and is therefore true for *all* values of x.

IV.

1. $4x^2+3y^2$.

2. $a^2(2x-3a)^2$ shillings.

3. $10x$ years.

4. Since the . . . . pounds which they gained were to be divided equally among x persons, each person's share must have been . . . . pounds, which $= x^3-x^2+5x+2$ pounds.

5. His present age is x years, suppose; then his age 5 years ago must have been . . . . . years, and his age 5 years hence will be . . . . . . years. And since we are told that his age 5 years ago was $\frac{5}{7}$ of what his age will be 5 years hence, we have the equation

. . . . = ...

from which we get $x = 30$.

6. Since he bought the goods for . . . pounds and sold the whole for . . . . pounds, his profit on the whole must have been . . . . . . pounds. But we are told that his profit on the whole was . . . . . . pounds. This gives us the equation

. . . . . = ...

from which we get $x = a+1$.

V.

1. $ax+by-c$ hours.

2. $ax-c$ miles.

3. The distance which he walks in the first a^2-x^2 hours is . . . . . . miles; and since he is to walk the remaining . . . . miles of the journey in . . . . . hours, he must during this time walk at the rate of . . . . . . miles an hour, which $= a^2-x^2$ miles an hour.

4. He started at the rate of x miles an hour, suppose. Then, since at this rate he could travel the whole distance in 10 hours, the whole distance must be . . . . miles, and half the distance must be . . . . miles. But since he performed the second half of the distance in . . . . hours, at the

rate of . . . . miles an hour, half the distance must be . . . . miles. These two algebraical expressions for half the distance give us the equation

$$\ldots\ldots = \ldots$$

from which we get $x = 6$.

He started therefore at the rate of 6 miles an hour, and the whole distance which he performed was 60 miles.

VI.

1. 20.
2. 13, 22.
3. 10, $2\frac{2}{3}$.
4. $\frac{1}{20}$, $\frac{57}{40}$, $\frac{1}{10}$, $\frac{63}{40}$.
5. 8, 12.

6. In the first three hours he walked altogether . . . . . miles, and in the remaining three-quarters of an hour he walked . . . . . miles; the whole distance which he walked in $3\frac{3}{4}$ hours must therefore be . . . . miles. But we are told that this whole distance was 13 miles. This gives us the equation

$$\ldots\ldots = 13,$$

from which we get $x = 4$.

During the first 3 hours, therefore, he walked at the rate of $3\frac{1}{2}$ miles an hour, and for the next three-quarters of an hour at the rate of $3\frac{1}{3}$ miles an hour.

VII.

1. Since he bought the horse for . . . pounds and sold him for . . . pounds, his gain must have been . . . pounds. But we are told that his gain was . . . pounds. Therefore

$$\ldots = \ldots$$

an equation from which we get $x = b + 1$.

2. Since the bill amounted to . . . shillings, the change out of a five-pound note must have been . . . . shillings. But, by hypothesis, the change out of a five-pound note

was . . . . shillings. These two algebraical expressions for the same thing give us the equation

. . . . . = ...

from which we get $x = 25$.

3. Since A bought his for . . . shillings and sold it for . . . shillings, his *gain* must have been . . . . . shillings. And since B bought his for . . . shillings and sold it for . . . shillings, his *loss* must have been . . . . . shillings. But, by hypothesis, A gained just half as much as B lost. This gives us the equation

. . . . . . = ...

from which we get $x = \frac{2m+n}{3}$.

4. The line is 2 feet long, and the two parts are $10\frac{1}{2}$ inches and $13\frac{1}{2}$ inches, respectively.

5. Since the algebraical quotient is . . . . . and the numerical quotient is 24, we have the equation

. . . . = 24

from which we get $x =$ . . .

The numbers are therefore 624 and 26.

VIII.

1. 1*l.* 10*s.*
2. $x-a$ pounds.
3. The distance which he walked in the first $a+x$ hours was . . . . miles, and the distance which he walked in the remaining . . . . . hours was . . . . . miles. The whole distance between A and B must therefore be . . . . . miles, which, when simplified, becomes $2a^3$ miles.
4. $a^3+b^3+c^3-3abc$ square feet.
5. x^2+xy+y^2 feet.
6. The two independent algebraical expressions for the person's distance south of B give us the equation

. . . . . = ...

from which we get $x = -1$. Since x denoted the person's position *north* of A, this negative value indicates that he was *one mile* to the *south* of A, and *not* to the *north* as the boy had supposed.

Again; since the person was one mile to the south of A and . . . . miles to the south of B, the distance between A and B must be 3 miles.

IX.

1. $6x(x-1)$ shillings.
2. 2 articles.
3. One pound.
4. The first lot of articles cost him . . . . shillings and the next lot cost him . . . . shillings, so that all the articles must have cost him . . . . . . shillings. But since he received . . . shillings' change out of the sovereign which he gave in payment, the total cost must have been . . . shillings. We have thus an algebraical expression and a numerical expression for the total cost, by equating which we get $x = 20$. Substituting this value of x in the other algebraical expressions, we find that he bought 5 articles at a shilling each, and 16 articles at 9*d.* each.
5. $5-x^2$ pence.
6. G.C.M. is $x-1$; L.C.M. is $2x(25x^2-4)(x^2-1)$.

X.

1. $\frac{x^4}{4}-\frac{3x^3}{4}-\frac{7x^2}{16}+\frac{3x}{2}+1$ square inches.
2. $3x^2-x+\frac{3}{2}$ feet.
3. $b+c-a$ hours.
4. $\frac{a^4}{a^2-x^2}$ days.
5. $6x(4x^2-1)^2$.

XI.

1. $a-7$.
2. $3x^2-2$.
3. $\frac{1}{a-b}$.

4. $\dfrac{2x^2}{(x-a)(x^2-b^2)}$.

5. $\dfrac{2xy}{x^2-y^2}$.

6. $2x^2-x-3$.

7. They were required to divide the fraction x by the fraction y, suppose. Some performed the operation indicated by $x \div y$, and others the operation indicated by $x-y$, and all got the same result, $4\frac{1}{2}$. We have, therefore, the two equations

. . . . = ... = . . .

from which we get $x = 5\frac{11}{14}$, and $y = 1\frac{2}{7}$.

The operation required to be performed was therefore $5\frac{11}{14} \div 1\frac{2}{7}$.

XII.

1. $x-x^2$.

2. The distance which he travels in the first a^2+ax+x^2 hours is . . . . . miles, and since the whole distance to be travelled is . . . . miles, the portion of the journey which still remains is . . . . miles, which, being simplified, is a^3-ax^2 miles.

3. $\dfrac{2ab}{a^2-b^2}$ pounds.

4. Let x denote the fraction of which the boy was to find $\frac{3}{4}$. The algebraical expression for the true answer is . . . ., while the expression for the boy's answer is . . . .; and since the boy's answer exceeded the true answer by . . ., we have the equation

. . . . . . = ...

from which we get $x = \frac{5}{6}$.

The boy's answer was, therefore, $\frac{10}{9}$, while the true answer is $\frac{5}{8}$.

XIII.

1. The x sovereigns = . . . shillings, the $\dfrac{2x}{5}$ sixpences

$= \ldots$ shillings, and there were $\frac{3x}{10}$ shillings besides; the whole sum must, therefore, have been . . . . shillings. But, by supposition, the whole sum was . . . shillings. This gives us the equation

$$\ldots\ldots = \ldots$$

from which we get $x = 10$, $\frac{3x}{10} = 3$, $\frac{2x}{5} = 4$.

2. Since the whole sum amounted to 205 shillings, and the algebraical expression for the whole sum expressed in shillings is . . . ., we have the equation

$$\ldots\ldots\ldots = 205$$

from which we get $x = \ldots$ Substituting this value of x in the algebraical expression for the number of half-crowns, we find the number of half-crowns to be 10.

3. If we suppose that the purse contained x sovereigns, ... half-crowns, and . . . . shillings, the whole sum must have amounted to . . . . . . shillings. But we are told that the whole sum amounted to 53 shillings. Therefore

$$\ldots\ldots = 53,$$

an equation from which we get $x = \ldots$ The purse contained, therefore, 2 sovereigns, 4 half-crowns, and 3 shillings.

4. The value of a sheep is x pounds, suppose. Then, since A takes . . . sheep and receives . . . pounds from B, the value of his share must be . . . . . . pounds. And since B takes . . . sheep and gives A . . . pounds, the value of his share must be . . . pounds. The two shares being supposed to be equal, we have the equation

$$\ldots\ldots = \ldots\ldots$$

from which we get $x = 3\frac{1}{2}$.

The value of a sheep is, therefore, 3*l.* 10*s.*

4. Another solution (by arithmetic).

The number of sheep altogether = ...; of these . . . are due to A; but he only takes . . . sheep, as he receives . . . pounds from B as an equivalent for the remaining . . .

sheep. The value of a sheep must therefore be .... pounds, which = 3*l*. 10*s*.

XIV.

1. Since he bought the goods for .... pounds, and sold them for .... pounds, his gain must have been .... pounds. But, by supposition, his gain was $4x$ pounds. Therefore

$$\ldots\ldots\ldots = 4x$$

an equation from which we get $x = \frac{mab}{(a^2-b^2)^2}$.

2. In a^4-b^4 hours A will have travelled ..... miles and B will have travelled .... miles, and then B will be .... miles in advance of A, which, being simplified, is $2a^3b^2$ miles.

3. Let x denote the true quotient. Now, to move the decimal point in a decimal fraction one place to the right is equivalent to multiplying the decimal fraction by .... The boy's answer therefore = ...; and since this exceeded the true answer by ..., we have the equation

$$\ldots\ldots = \ldots\ldots$$

from which we get $x = \frac{1}{4}$ or ·25.

4. $\frac{x}{2}-3$ feet.

XV.

1. In $3(a+b-x)$ years.
2. $x^2+xy+x+y^2-y+1$ pounds.
3. $x^2+2xy+3y^2$ coins.
4. $(a+x)^3$ days.
5. $\frac{x}{1+x}$ of the work.
6. $\frac{2ax}{x^2-a^2}$ pounds.

7. Let the required quantity be x lbs. Then, since the value of these x lbs. at ... pence per lb. is .... pence, and

the value of the m lbs. at b pence per lb. is . . . . pence, the value of the whole mixture must be . . . . . pence. Again, since the mixture is to contain . . . . lbs., and is to be worth c pence per lb., the value of the whole mixture must be . . . . pence. These two algebraical expressions for the value of the whole mixture give us the equation

. . . . . . = ...

from which we get $x = \dfrac{m(c-b)}{a-c}$.

XVI.

1. $\dfrac{x}{a}$ hours.

2. $\dfrac{4}{3(x+1)}$ days.

3. It travelled the first x miles in . . . . hours, and the next y miles in . . . hours, so that it must have travelled the whole distance in . . . . . . hours. But we are told that it travelled the whole distance in $2a$ hours. This gives us the equation

. . . . . . $= 2a$.

Again; since the whole distance is . . . . miles, if the train had travelled throughout at the rate of a^2+b^2 miles an hour, it must have taken it . . . . hours to accomplish the whole distance. But we are told that in this case it would have accomplished the whole distance in $\dfrac{x-y}{2ab}$ hours. This gives us another equation—

. . . . . $= \dfrac{x-y}{2ab}$;

and from these two equations we get $x = (a+b)^2$; $y = (a-b)^2$.

4. Since the length is . . . . yards and the breadth is . . . . yards, the area must be . . . . square yards. But we are told that the area is 15 square yards. Therefore

$$\ldots\ldots = 15,$$

an equation from which we get $x = \pm 4$.

The length is therefore 5 yards, and the breadth is 3 yards.

5. Let us suppose that there were in all x candidates. Then, since there were 24 places (or centres), the average number of candidates per centre must have been .... If London and its 95 candidates be omitted, the total number of candidates will only be ...., and the number of centres will be ..., so that the average number of candidates per centre must be ....; and since we are told that the average per centre is in this case $2\frac{17}{24}$ less than before, we have the equation

$$\ldots\ldots = \ldots\ldots$$

from which we get $x = 785$.

6. $2a(x-a)^2(x+a)$.

XVII.

1. ax^2-bx-c hours.

2. The distance which he travels altogether in the first c hours is .... miles, and since he is to accomplish the remaining ...... miles of the journey in .... hours, he must travel at the rate of ...... miles an hour, which, being simplified, is $a+b-c$ miles an hour.

3. Since the one walks at the rate of x miles an hour, the distance which he walks altogether must be .... miles; and since the other walks at the rate of .... miles an hour, the distance which he walks altogether must be .... miles. The distance between A and B must therefore be .... miles, which, being simplified, is $a(a^2+ax+x^2)$ miles.

4. They will meet in x hours, suppose. Then, since the former walks at the rate of ... miles an hour, and the other at the rate of ... miles an hour, when they meet, the one will have walked .... miles, and the other .... miles. The distance between the towns must therefore be .....

miles. But we are told that the distance between the towns is 60 miles. This gives us the equation

. . . . = 60

from which we get $x = 8$. They will therefore meet in 8 hours at a distance of . . . miles from A and . . . miles from B.

Another Solution.

They will meet at x miles from A, suppose; and . . . . miles from B. Now, since the one walks altogether x miles, at the rate of . . . miles an hour, it must take him . . . hours to do so; and since the other walks altogether . . . . miles at the rate of . . . miles an hour, it must take him . . . hours to do so. And since they started at the same time, we have the equation

. . . . = ...

from which we get $x = 32$.

XVIII.

1. $\frac{x+2}{x-2}$ pounds.

2. The algebraical expression for the number of shillings which each person received is . . . ., which when simplified becomes . . . . ., and since we are told that each person received 2 shillings, we have the equation

. . . . = 2,

from which we get $x = 12$.

3. $x+y-z$ miles per minute.

4. The total sum contributed by the three divisions was . . . . pounds, and the total number of boys in the three divisions was . . . . Each boy must therefore have contributed on an average . . . . pounds, which, being simplified, is $b+c$ pounds.

XIX.

1. A had $a-5b+c$ pounds; and B had $7a+b-3c+4d$ pounds.

2. After the game, A, since he gained, had . . . . . pounds; and B, since he lost, had . . . . pounds; and, since we are told that A's money was then 5 times as much as B's, we have the equation

. . . . . = ...

from which we get $x = \frac{2}{3}(a-b)$.

3. 20 days.

4. The fraction of the work which A performed is x; the fraction of it which B performed is . . . .; and the fraction of it which C performed is $\frac{3}{10}$; and these three fractions make up the whole work, so that we have the equation

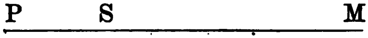
. . . . = 1,

from which we get $x = \frac{1}{2}$. A therefore performed one-half of the whole work, and B performed one-fifth of it.

5. The G.C.M. is $a^2 + b^2$; and the
L.C.M. is $(a^2+b^2)(2a^2-6ab+3b^2)(3a^2-6ab+2b^2)$.

XX.

1. $\frac{2}{3}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$.
2. $x = 4, y = 1$.
3. $x = 3, y = 2$.
4. Let P and S be the positions of the privateer and ship respectively when the chase begins; and let M be the place at which the privateer overtakes the ship,



Then, since the privateer runs at the rate of x miles an hour, in $10\frac{1}{2}$ hours it will have run . . . . miles, which is therefore the algebraical expression for the distance P M.

And since the ship sails at the rate of $\frac{4x}{5}$ miles per hour, in $10\frac{1}{2}$ hours it will have sailed . . . . miles, which is therefore the expression for the distance S M.

And the difference between these two distances is P S, which we are told is 25 miles. Therefore

. . . . = ...

From this equation we get $x = 11\frac{19}{21}$.

XXI.

1. In one day A alone can do . . . of the work, B alone can do . . . . of it, and C alone can do . . . . of it. The three together therefore in one day can do . . . . of the work, which, being simplified, is $\frac{a+2x}{x(a^2-x^2)}$ of the work.

2. $\frac{2}{3}x^4-2x^3+\frac{7}{3}x^2-\frac{5}{4}x-1$ square feet.

3. $\frac{x^2+2x-2}{x+4}$ feet.

4. $\frac{a+x}{ax}$ pounds.

5. $\frac{9x^2}{4}-\frac{9ax}{2}-\frac{18x}{b}+\frac{9a^2}{4}+\frac{18a}{b}+\frac{36}{b^2}$ square feet.

6. The man alone would consume it in x days, suppose. Now, since the man alone would consume it in x days, and the woman alone consumed it in 30 days, in one day the man alone could consume . . . of it, and the woman alone could consume . . . of it; hence, in one day both together could consume . . . . . . of it. But since they consumed it together in 12 days, in one day they could consume . . . of it. Therefore

. . . . = . . .

an equation from which we get $x=20$.

XXII.

1. In one day A can do . . . . . . of the work, and in one day B can do . . . . of it. Therefore, in one day A and B together could do . . . . of the work, and in c days they could do . . . . of it, which, being simplified, is $\frac{2x-a-b}{(x-a)(x-b)(a-b)}$ of it.

2. $\frac{2x^2-3}{2x+3}$ days.

3. The whole sum contributed by the $a+2b$ persons was . . . . shillings. The whole sum received by the $c-3d$ persons was . . . . . shillings. Consequently, the sum

to be divided equally among $a+2b+c-3d$ persons was .... shillings, and of this sum each person received ...... shillings, which, being simplified, is $a+2b-c+3d$ shillings.

4. $9a^2-x^2$ square feet.

XXIII.

1. The L.C.M. of the four expressions is $2\times 3^2\times 7a^3(x-3)(x+3)^2(x^2-3x+9)$; the G.C.M. of the first two expressions is $14a$; and the G.C.M. of the last three expressions is $7(x+3)$.

2. The $\frac{2x}{3}$ persons received altogether .... pence; the 4 persons received altogether ... pence; and the remaining .... persons received altogether .... pence. The whole sum received by the x persons must therefore have been ..... pence, which $=\frac{20x}{3}+4$ pence.

3. There were x persons altogether, suppose. Then the ... persons who received 9 pence each received altogether ... pence or ... shillings; the ... persons who received 3 shillings each received altogether .... shillings; the two persons who received a shilling each received altogether 2 shillings; and the remaining .... persons who received 1*s*. 6*d*. each received altogether .... shillings. The whole sum received by the x persons must therefore have been ...... shillings. But we are told that the whole sum was a sovereign, or 20 shillings; therefore

$$\ldots\ldots = 20,$$

from which we get $x=18$.

4. The first rectangle is x inches long, suppose, and .... inches broad: its area must therefore be .... square inches. The other rectangle being .... inches long and .... inches broad, its area must be ..... square inches. And since we are told that the area of the first rectangle is $\frac{3}{4}$ of the area of the other, we have the equation

$$\ldots\ldots = \ldots\ldots$$

from which we get $x=10$ or 18.

The lengths of the two rectangles are therefore 10 inches and 20 inches, respectively, and the breadths 6 inches and 4 inches; or else, the lengths are 18 inches and 28 inches, and the breadths 14 inches and 12 inches. For either value of x will satisfy the conditions of the problem.

XXIV.

1. The area to be covered being . . . . square feet, and the width of the paper being . . . . . feet, the length must be . . . . . feet, which, being simplified, is a^3-4ax^2 feet.

2. $2(a^3+x^3)$ square feet.

3. $6x^4-\frac{26}{3}a^2x^2+\frac{3}{2}a^4$ shillings.

4. The area of each square being . . . . square inches, and the area of the rectangle being . . . . square feet, or . . . . . square inches, the required number of squares must be . . . . ., which $=\dfrac{144(z+x+y)}{z-x-y}$.

5. Let us suppose that the piece was x yards long; then, since the width of it was . . . of a yard, its area must have been . . . . square yards. At the rate of 6 shillings per *yard* the upholsterer must have sold the whole piece for . . . shillings, while at 6 shillings per *square yard* he would have sold it for . . . . shillings. And since we are told that at the former rate he gained . . . shillings more than he would have gained at the latter rate, we have the equation

. . . . = . . .

from which we get $x=20$.

XXV.

1. After the game, A had $\dfrac{2(a^4+x^4)}{a^4-x^4}$ shillings, and B had $\dfrac{4a^2x^2}{a^4-x^4}$ shillings.

2. Since A began the second game with a sum of . . . . shillings, and B with a sum of . . . . shillings, after this game A must have had . . . . . . shillings, which, being

simplified, is $\frac{2(a^2-x^2)}{a^2+x^2}$ shillings; and B must have had .... shillings, which, being simplified, is $\frac{8a^2x^2}{a^4-x^4}$ shillings.

3. 3*l*. 9*s*. 2*d*.

4. Let us suppose that A had x pounds, and that B had y pounds, when they began playing. Then, since in the first game B won ... pounds from A, at the end of this game A must have had .... pounds, and B must have had .... pounds. But we are told that at the end of this game A's money was $\frac{3}{4}$ of B's. This gives us the equation

.... = ...

Again; since A began the second game with a sum of .... pounds, and B with a sum of .... pounds, and in the game, A won .... pounds from B, at the end of it A must have had .... pounds, and B must have had .... pounds. But we are told that at the end of the second game A's money was to B's as 5 is to 4. This gives us the equation

..... = ...

From this and the preceding equation we get

$$x = \frac{27b}{8}+a, \text{ and } y = \frac{9b}{2}-a.$$

XXVI.

1. G.C.M. $= 2ax(10x^2-17x+3)$
L.C.M. $= 12a^2x^2(10x^2-17x+3)(2x-3)(5x-1)$,

which, when resolved into its elementary factors, is

$$2^2\times 3a^2x^2(2x-3)^2(5x-1)^2.$$

2. $\frac{12(2x-y)}{xy}$.

3. Let y denote the required time in days. Then, since the time must be in inverse proportion to the number of men, we have the equation

$$\frac{x-\frac{2a}{3}}{3ax-2a^2}=\frac{y}{4},$$

from which we get $y=\frac{4}{3a}$.

The second answer is evidently $\frac{1}{a}$ days.

4. If he had had twopence more, he would have had altogether . . . . pence; and since this sum would have been divided among . . . . beggars, each beggar would have received . . . . pence, which, being simplified, is . . . . pence. But we are told that in this case each beggar would have received 8 pence. This gives us the equation,

$$\ldots\ldots = 8,$$

from which we get $x=6$. Each beggar therefore received $7\frac{1}{2}d$.

XXVII.

1. $2b$ pounds.
2. Neither a loss nor gain.
3. $\frac{2ab}{a^2-b^2}$ pence.
4. Substitute for x the value given in the answer; and let y denote the missing denominator. Solving the equation, then, with regard to y, we shall find $y=a+b$.

XXVIII.

1. From the resulting equation we shall get $x=-5$. The gain on the whole is therefore -2 pounds, which means that the *loss* on the whole is $2l.$, and, consequently, the loss per article must be $4\frac{4}{5}$ pence.

2. The numerator vanishes since $x-7$ is a factor of it, while the numerical value of the denominator is 149. The required value, therefore, is zero.

3. The two fractions are identical, as may be seen by dividing the numerator and denominator of the first fraction by $x-1$. When $x=1$ the first fraction assumes the

form $\frac{0}{0}$, while the second fraction becomes 4. By this we must understand that 4 is the *limit* to which the first fraction approaches when the numerical value of x is made more and more nearly equal to 1. For example, if $x = 1{\cdot}000002$ the numerical value of the first fraction is *a very little less than* 4, while if $x = {\cdot}99999$ its numerical value is *a very little greater than* 4, as will be found upon trial.

4. G.C.M. $= (2x-3)^2$; L.C.M. $= (2x-3)^2(x^4-1)$.

XXIX.

1. $\dfrac{4a^3x^3}{a^6x^6-1}$ miles per hour; $\dfrac{2(a^6x^6+1}{a^6x^6-1}$ miles per hour.

2. In still water he would row at the rate of . . . . miles an hour, and therefore *against* the current he would row at the rate of . . . . miles an hour, which, being simplified, is $\dfrac{4xy}{x^2-y^2}$ miles an hour.

3. The rate at which they would go down the stream with the current to help them would be . . . . miles, and they would therefore accomplish the whole distance down, viz., a^3-x^3 miles, in . . . . . hours, which, being simplified, is $a-x$ hours.

Similarly it may be shown that they would row the a^3+x^3 miles up in $a+x$ hours.

4. In still water he could row at the rate of x miles an hour, suppose. Since the river has a current of one mile per hour, he must have gone down at the rate of . . . . miles per hour, and up at the rate of . . . . miles per hour. He rowed, therefore, the whole distance down, viz., ... miles, in . . . . hours, and the same distance up in . . . . hours, thus accomplishing the whole journey down and up in . . . . . . hours. But we are told that he performed the whole distance down and up in n hours. This gives us the equation,

$$\ldots\ldots = n,$$

from which we get $x = \dfrac{m}{n} \pm \dfrac{\sqrt{(m^2+n^2)}}{n}$.

It is evident that the *lower* sign would give a *negative* value to x, so that the conditions of the problem are satisfied only by taking the *upper* sign.

XXX.

1. The current flows at the rate of x miles an hour, suppose. Now, since they rowed down a distance of a miles in m hours, the rate at which they went down must have been . . . miles an hour. In still water therefore they could row at the rate of . . . . miles an hour. Again; since they rowed a miles up the river in n hours, the rate at which they went up must have been . . . miles an hour. In still water, therefore, they could row at the rate of . . . . miles an hour. These two algebraical expressions for the rate at which they could row in still water give us the equation,

. . . . = ...

from which we get $x = \frac{a(n-m)}{2mn}$.

2. In still water they could row at the rate of x miles an hour, suppose. Now, since they can row down at the rate of . . . . miles an hour, and could row in still water at the rate of x miles an hour, the rate of the current must be . . . . miles an hour. Also, since they can row up at the rate of . . . . miles an hour, and could row in still water at the rate of x miles an hour, the rate of the current must be . . . . miles an hour. These two algebraical expressions for the rate of the current gives us the equation

. . . . . = ...

from which we get $x = \frac{2mn}{m-n}$.

3. At x miles from A, suppose, and, consequently, ... miles from B. Now, since the one who starts from A goes down at the rate of . . . . miles an hour, he accomplishes the whole x miles down to the meeting-place in . . . . hours. And, since the one who starts from B goes up at the rate of . . . . miles an hour, he accomplishes the whole . . . .

miles to the meeting-place in . . . . hours. And we are told that they started at the same time; we have, therefore, the equation

$$\ldots\ldots=\ldots\ldots$$

from which we get $x=\frac{p(m+n)}{2m}$.

They will meet, therefore, at a distance of $\frac{p(m+n)}{2m}$ miles from A, and $\frac{p(m-n)}{2m}$ miles from B.

XXXI.

1. $3+5x-x^2$.
2. $2+2x-3x^2$ miles per hour.
3. His gain on each article being . . . . pence, his gain on the whole must be . . . . . pence; and, since this sum was divided equally among . . . . persons, each person must have received . . . . pence, which, being simplified, is $8-6x-9x^2$ pence.
4. $\left(\frac{ax+b}{b}\right)^2$ hours.
5. In one hour A can perform . . . . of the work, and B in one hour can perform . . . . of it. Therefore, both working together can in one hour perform . . . . . of it, which, being simplified, is $\frac{2ab^2x}{(ax-b)(ax+b)^2}$ of it.
6. $\frac{4}{5}$.

XXXII.

1. G.C.M. $= ax$.
   L.C.M. $= a^3x^3(x-1)^2(x+1)^2(x^2+1)$.
2. $y=x(a+3b-c)$.
3. $x=3$.
4. Let x denote the required number. Then, since he bought the oranges at the rate of 12 oranges for 6 pence,

he must have bought the x oranges for . . . . pence. And, since he sold his oranges at the rate of 3 for 2*d.*, he must have sold the . . . . oranges which had not been spoiled for . . . . pence. His whole profit must consequently be . . . . pence. But we are told that his whole profit is 10*s.* 9*d.* or 129 pence. Therefore,

$$\ldots\ldots = 129,$$

an equation from which we get $x = 974$.

5. $a + x$ shillings.

6. The second will overtake the first at x minutes past 3, suppose. Now, since the first starts at a quarter-past 3, at x minutes past 3 he will have been travelling . . . . minutes; and the distance which he will have gone over in this time (since he goes at the rate of . . . . yards per minute) will be . . . . yards.

In the same way it may be shown that the distance which the second will have gone over at x minutes past 3 will be . . . . yards.

And, since by supposition the second will have overtaken the first at x minutes past 3, we have the equation

$$\ldots\ldots = \ldots\ldots$$

from which we get $x = 70$.

The second will therefore overtake the first at 70 minutes past 3; that is, at 10 minutes past 4; and at . . . . yards or . . . miles, from A.

Another Solution.

Let x denote the distance (in yards) from A, at which the second will overtake the first.

Now, since the first goes at the rate of . . . yards per minute, he will go over x yards in . . . . minutes.

And since the second goes at the rate of . . . yards per minute, he will go over x yards in . . . . minutes.

Also, since the first starts at a quarter-past 3, while the second does not start till 20 minutes past 3, the second goes over the x yards in . . . minutes less time than the first does. This gives us the equation

$$\ldots\ldots = \ldots\ldots;$$

from which we get $x = 22000$.

The distance being thus known, the time is easily found.

7. Let us suppose that A alone could do the work in x hours, and B alone in . . . . hours; it follows that A alone could in one hour do . . . of it, and that B alone in one hour could do . . . . of it; and, consequently, both working together, could in one hour perform . . . . . of it. But, since they can together perform the whole work in $3\frac{3}{4}$ hours, in one hour they can perform . . . of it. We have, therefore, the equation

$$\ldots\ldots = \ldots\ldots$$

from which we get $x = 10$ or $1\frac{1}{2}$, the first of which only satisfies the conditions of the problem.

XXXIII.

1. $7x - \frac{a}{3}$ shillings.
2. $3a + \frac{4x}{3}$ men.
3. $\frac{2bx}{a^2 - b^2}$ years; $\frac{2ax}{a^2 - b^2}$ years.
4. $x = \frac{2(a^2 + b^2)}{a^2 - b^2}$; each army contained at first $\frac{a^2 - ab + 2b^2}{a^2 - b^2}$ men.

5. Let x denote the greater fraction, and y the less. Then we shall get $x = \frac{33}{34}$, $y = \frac{15}{34}$.

XXXIV.

1. $x^2 - b - c$ books.
2. ab pounds.
3. 3 or 7 persons.

4. $y = \frac{3(x+3)^2}{2x+3}$.

5. $\frac{2x}{1+x}$.

6. If x denotes the larger fraction and y the less, we shall find $x = \frac{2ab}{b-a}$, $y = \frac{2ab}{b+a}$.

XXXV.

1. $\frac{16}{81}x^4 + \frac{4}{9}x^2 + 1$ miles.
2. $\frac{3}{2}x^2 - ax$ minutes.

3. In $\frac{x}{2} - 3$ minutes the former will have run ..... yards, and the other ..... yards; so that in this time the former will be ..... yards a-head of the latter, which, being simplified, is $\left(\frac{x}{2} - 3\right)^2$ yards.

Otherwise; since the former beats the latter every minute by .... yards, in $\frac{x}{2} - 3$ minutes he will beat him by $\left(\frac{x}{2} - 3\right)^2$ yards.

4. A man's share exceeded a woman's by $(a-x)\left(x - \frac{1}{x}\right)$ shillings.

5. Since the former travels at the rate of m miles an hour, in x hours he will have travelled ... miles; and since the other travels at the rate of n miles an hour, in x hours he will have travelled ... miles. From this, and since they are travelling in opposite directions, it follows that in x hours they will be separated by .... miles. But we are told that in x hours they will be separated by mn miles. This gives us the equation

$$\ldots\ldots = mn,$$

from which we get $x = \frac{mn}{m+n}$.

XXXVI.

1. 25 per cent.

2. $33\frac{1}{3}$.

3. I must gain $\frac{7}{100}$ of the prime cost; I must therefore sell it for $x+\frac{7x}{100}$ shillings, which $=\frac{107x}{100}$ shillings.

4. He loses $\frac{7}{100}$ of the prime cost; he must therefore sell the first article for $\frac{93x}{100}$ shillings, and the second article for $\frac{93}{100}\left(\frac{x}{3}-2\right)$ shillings.

5. If I gain, I gain .... pence on every a pence which I expend; and therefore on every 100 pence which I expend I gain $\frac{100}{a}(b-a)$; that is, I gain at the rate of $\frac{100}{a}(b-a)$ per cent.

But if I lose; since on every a pence which I expend I lose ...., pence, on every 100 pence which I expend I lose $\frac{100}{a}(a-b)$ pence; that is, I lose at the rate of $\frac{100}{a}(a-b)$ per cent.

6. Since the buying price is to the selling price as 100 is to 117, he must have bought the tea at the rate of $\frac{100}{117}\left(\frac{3x}{2}-3\right)$ shillings per lb.

7. $\frac{20}{3m}(7x-15m)$.

8. The loss per cwt. $=20a-\frac{x}{3}$ shillings; and the loss per cent. $=100-\frac{5x}{3a}$.

9. $\frac{109}{100}(3x-2y)$ shillings.

10. $\frac{91}{100}(3x-2y)$ shillings.

11. $\frac{100}{x}(y-x)$; $\frac{100}{y}(y-x)$.

XXXVII.

1. $24a^2x^2(x-a)(x^3+a^3)$; $4(x^2-ax+a^2)$.

2. $\dfrac{1}{x+1}$.

3. $2(x+1)l.$; $2l.$; $2(x^2-x-3)l.$

4. In one day A performed $\frac{x}{3}$ of the work, and in one day B performed $\dfrac{1}{3(1+x)}$ of it.

5. Let x denote the fraction of the work which A performs; then the remainder (or the fraction of it which B performs) will be denoted by . . . .

And, since A performed x of the work in b days, and B performed . . . . of it in $b+c$ days, in one day A performed . . . of it, and B in one day performed . . . . of it; so that in one day A and B together performed . . . . of it. But, since we are told that they could together do the whole in a days, in one day they could together do . . . of it. These two expressions for what they could together do in one day give us the equation

. . . . = . . .

from which we get $x=\dfrac{b(b+c-a)}{ac}$. This being the fraction of the work which A performed, the remainder (or the fraction of the work which B performed) must be . . . . ., which $=\dfrac{(a-b)(b+c)}{ac}$.

6. Let x denote the sum (in pounds) which the one who falls ill is entitled to receive. Then, since between them they earn $25l.$, the other is entitled to receive . . . . pounds. And since the first earns x pounds in . . . . days, in one day he earns . . . . pounds; and since the other earns the remaining . . . . pounds in . . . days, in one day he earns . . . . pounds. In one day, therefore, they can earn between them . . . . pounds. But, since we are told that they earn the whole $25l.$ in 75 days, in one day they could earn . . . pounds. These two expressions for what they could together earn in one day give us the equation

. . . . . = . . .

from which we get $x = 6$. The one who falls ill is therefore entitled to receive 6*l*., and the remaining 19*l*. is due to the other.

Another Solution.

Taking the whole work as unity, let x denote the fraction of it which the one who falls ill performs. Then, since the whole work is denoted by 1, the fraction of it which the other performs must equal . . . . And since the first performs x of the work in . . . days, in one day he performs . . . of it; and since the other performs the remaining . . . . of it in . . . days, in one day he performs . . . . of it. Consequently, the portion of the work performed by both together in one day equals . . . . But, since they could together perform the whole work in 75 days, in one day they could do . . . of it. Therefore,

$$\ldots. = \ldots$$

an equation from which we get $x = \frac{6}{25}$.

And since the first performs $\frac{6}{25}$ of the work, he is entitled to $\frac{6}{25}$ of the 25*l*., that is, to 6*l*.; and the remaining 19*l*. is due to the other.

Solution by Arithmetic.

Since in 75 days they could together earn 25*l*., in the . . . days during which they worked together they must have earned . . . . pounds. And, since the one who continued the work afterwards earned the remaining . . . . pounds in 50 days, it follows that in the whole 95 days during which he worked he must have earned . . . . pounds, that is, 19*l*. And his companion earned the remaining 6*l*.

7. $\frac{b}{a}$ days.

8. $\frac{a}{x}$ of it.

9. In one day he could do . . . . of it, and he could do the whole in $\frac{ac}{a-b}$ days.

XXXVIII.

1. $x^2+y^2+z^2$ shillings.

2. The x lbs. cost him altogether . . . . shillings, and the y lbs. cost him altogether . . . . shillings, so that the whole mixture must have cost him . . . . shillings. And since the whole mixture contained . . . . lbs., and was sold at b shillings per lb., the whole must have been sold for . . . . shillings. The gain on the whole must therefore have been . . . . shillings, which $= (x+y)(b-a)$ shillings.

Each letter being understood to denote a positive quantity, if a is numerically greater than b this expression for the gain will be negative, which indicates that he *lost* $(x+y)(a-b)$ shillings.

3. Let x denote the required number of gallons. Then, since the value of the x gallons of the first sort of wine is . . . shillings, and the value of the $a-2$ gallons of the other sort is . . . . shillings, the value of the mixture must be . . . . . shillings. But we are told that the value of the mixture is to be . . . . . shillings. These two expressions for the value of the mixture give us the equation

$$\ldots\ldots = \ldots\ldots$$

from which we get $x = \dfrac{4}{a-2}$.

4. He takes x quarts from the first sort of wine, suppose. Then, since the mixture is to contain . . . quarts, he must take . . . . quarts from the second sort. And, since the value of the x quarts which he takes from the first sort is . . . . shillings, and the value of the . . . . quarts which he takes from the other sort is . . . . shillings, the value of the whole mixture must be . . . . shillings. But, since the mixture is to contain . . . quarts, and to be worth . . . shillings per quart, the value of the mixture must also be . . . shillings. These two expressions for the value of the mixture give us the equation

$$\ldots\ldots = \ldots\ldots$$

from which we get $x = \dfrac{n(c-b)}{a-b}$.

5. The equation is satisfied by any one of the following three values of x; 0, 6, $\frac{3}{2}$.

XXXIX.

1. $\frac{a}{a+b}$ and $\frac{b}{a+b}$.

2. $\frac{2}{2a+1}$.

3. The water being . . . of the whole mixture, in $4a^2-b^2$ quarts of the mixture there will be . . . . quarts of water, which $=\frac{(2a-b)^2(2a+b)}{4a}$ quarts; and $\frac{(2a+b)^2(2a-b)}{4a}$ quarts of wine.

4. The quantity taken from the cask A contained, we will suppose, x gallons of wine and y gallons of water. Then, since in the mixture to be made there are to be b gallons of wine and c gallons of water, the quantity taken from the cask B must contain . . . . gallons of wine, and . . . . gallons of water.

Now, in the quantity taken from the cask A, the wine must have the same proportion to the water as in the cask; this gives us the equation

. . . . . = ...

And since the same remark applies to the quantity taken from the cask B, we have also the equation

. . . . = ...

From these two equations we get $x=\frac{m(pc-qb)}{pn-qm}$ and $y=\frac{n(pc-qb)}{pn-qm}$; and consequently, $x+y$, the number of gallons taken altogether from the cask A must $=\frac{(m+n)(pc-qb)}{pn-qm}$; and consequently also, the number of gallons taken altogether from the cask B must $=\frac{(p+q)(bn-cm)}{pn-qm}$.

Another Solution.

If we suppose x gallons to be taken from the cask A;

then, since the mixture is to contain altogether . . . . gallons, we must take . . . . gallons from the cask B. Of the x gallons taken from the cask A, ... gallons will be wine and . . . . gallons will be water; and of the . . . . gallons taken from the cask B, ... gallons will be wine and . . . . gallons will be water; consequently, the mixture to be made from the two casks will contain . . . . gallons of wine and . . . . gallons of water. But we are told that this mixture is to contain b gallons of wine and c gallons of water. This gives us a choice of two equations:

$$\ldots\ldots = b,$$

$$\text{and } \ldots\ldots = c,$$

from either of which we shall get $x = \dfrac{(m+n(pc-qb)}{pn-qm}$, which is therefore the number of gallons taken from the cask A; and from this it is easy to see that the number of gallons taken from the cask B is $\dfrac{(p+q)(bn-cm)}{pn-qm}$.

5. $73(2x-3)$.

XL.

1. $\dfrac{1+x^2}{x(1+x)^2}$ shillings; $\dfrac{(1+x^2)^2}{x^2(1+x)^2}$ shillings.
2. $\dfrac{(2a-x)^2}{10(2a+x)}$ days.
3. $\dfrac{3x^2}{4} - 2x + \frac{4}{3}$ shillings.
4. $c-a-b$ hours.
5. $\dfrac{b(1+a^2)}{2}$ feet.

6. Let x denote the length in feet. Then, since the breadth is to the length as 2 is to 3, the breadth must be . . . feet, and the area must be . . . . . square feet. Hence, the cost of the field at 9*d.* (or $\frac{3}{4}$*s.*) per square foot must be . . . . . . shillings. But we are told that the cost is 62*l.* 10*s.* These two expressions for the cost of the field give us the equation

. . . . . = . . .

from which we get $x = 50$. The length is, therefore, 50 feet and the breadth $33\frac{1}{3}$ feet.

XLI.

1. G.C.M. $= 2x(5x-1)$; L.C.M. $= 2ax(5x-1)(9x^4-25)$.
2. $c+b-2a$ yards; $abc(b-c-2a)$ pounds.
3. $\sqrt{\left(\frac{16a^4}{81}-x^4\right)}$ square yards.
4. $x = a$.
5. $x = \frac{4}{3}$; length $= \sqrt{2}$ feet; breadth $= \sqrt{\frac{2}{3}}$ of a foot.
6. $x = 2a$; $x+2a = 4a$.

XLII.

1. $x+a-2\sqrt{ax}$ square inches.
2. $x-a$ square inches.
3. $\sqrt{(x^2-4)}$ square feet.
4. $x = 2a$; length $= 2\sqrt{a}$ yards; breadth $= \sqrt{a}$ yards.
5. $6x\sqrt{a}-a-9a^2$ square feet.
6. $x = \sqrt{a}$.
7. $\sqrt{(a+b+c)}$ inches.

XLIII.

1. $\sqrt{(x^2+a^2)}$ inches.
2. $x-3$ feet.
3. Since the area of the first square is . . . . square yards, the length of its side must be . . . . . yards. For a similar reason the side of the second square must be . . . . yards. And we are told that the side of the first square is . . . yards shorter than the side of the other. We have therefore the equation

. . . . . = ...

from which we get $x = 1$. The side of the first square is, therefore, 2 yards, and the side of the other 4 yards.

4. Suppose the side of the square to be x feet. Then

the breadth of the rectangle must be .... feet, and the length of the rectangle must be .... feet, and the area of the rectangle must consequently be .... square feet. This being by supposition equal to the area of the square, we have the equation

$$.....=...$$

from which we get $x = 10$. The breadth of the rectangle is therefore 4 feet, and the length is 25 feet.

5. $x = 2a$. The area of the rectangle or square is therefore $2a^2$ square feet.

XLIV.

1. $2x-3$.
2. $2ax$.
3. The three sides are respectively 21, 28, and 35 inches; and the area is 294 square inches.
4. $\dfrac{4ax}{a-x}$ shillings.
5. G.C.M. $= 2(x^2+1)$;
   L.C.M. $= 12(x^2+1)(x^2-x+1)(x^2-x-1)$.

XLV.

1. $a-b+c$ inches.
2. $x = 9$.
3. $\sqrt{x}-\sqrt{3}$ yards.
4. $x = 9a^2$.
5. The first rectangle is x inches wide, say; then its length must be .... inches, and its area must be .... square inches. The second rectangle being .... inches long and .... inches wide, its area must be .... square inches. And the two rectangles being equal by supposition we have the equation

$$......=.....$$

from which we find the length of the first rectangle to be 10 inches, and the length of the other 20 inches.

6. The second exceeds the first by $\dfrac{2abc^2}{(a+b)^2-c^2}$ that is, by

$$\frac{2abc^2}{(a+b+c)(a+b-c)}.$$

XLVI.

(1) $\dfrac{(x-a)^4}{x^2+ax+a^2}$.

(2) $\dfrac{a+b}{a-b}$.

(3) $\dfrac{2b(a-x)}{(a-b)^2(a+b)}$.

(4) $\dfrac{a^2}{(a+b)^2(a-b)}$.

(5) $\dfrac{(a+b)(1-a+b)}{a^2+ab+b^2}$.

(6) $\dfrac{2x(a-x)}{(a+x)^2}$.

(7) $-(a-x)^2$.

XLVII.

1. 0; 6; $\frac{1}{6}$; $\frac{1}{2}$; and -1.

2. This being an equation of *identity*, any value of x will satisfy it.

3. When $x=2$, the first expression assumes the form $\frac{0}{0}$, while the second becomes equal to 4. We are, therefore, to understand that 4 is the *limit* to which the first expression approaches, as x approaches the limit 2. For example, if $x=1{\cdot}99999$, the numerator and denominator of the first expression will very nearly vanish, but not quite; and the numerical value of the first, as well as of the second, expression will be 3·99999.

4. $\dfrac{48(6-x)}{(x-4)(x+3)(x-2)}l$. When $x=8$, this becomes $-\frac{4}{11}l$, which indicates that, instead of gaining, he *loses* $\frac{4}{11}l$.

5. $\dfrac{2(6-x)}{(x-2)(x-3)}$ miles. When $x=8$, this becomes $-\frac{2}{15}$ miles; and $-\frac{2}{15}$ miles to the *east* of A *means* $\frac{2}{15}$ miles to the *west* of A.

XLVIII.

1. Suppose the stock in trade to be worth x pounds. When they dissolve partnership, since the value of what A

takes in stock is . . . . pounds, while he also receives . . . . pounds from B, the value of what A receives altogether in stock and money is . . . . pounds. But since the whole stock is worth x pounds, and A's share of this is twice as much as B's share, the value of A's share must also be . . . . pounds. These two expressions for the value of A's share give us the equation

. . . . . = ...

from which we get $x = 1200$.

2. 15 gallons.

3. He played x games, suppose, with B, out of which he won . . . . games; and he played . . . . games with C, out of which he won . . . . games. And since we are told that he won altogether . . . . games, we have the equation

. . . . = . . .

from which we get $x = 12$.

4. Let us suppose that each candidate received on an average x marks; then the total number of marks received by the 40 candidates will be . . . . If this one candidate and his 120 marks had been left out of account, the total number of marks would have been . . . . ., and the average number of marks for each candidate would have been . . . . .; and since we are told that this is . . . marks higher than the actual average for each candidate, we have the equation

. . . . . . = ...

from which we get $x = 900$.

XLIX.

1. $100x+10y+z$; $100z+10y+x$.
2. $10^3x+10^2y+10m+n$; $10^3m+10^2n+10x+y$.
3. $10m+n+\frac{x}{10}+\frac{y}{10^2}+\frac{z}{10^3}$.
4. $8^3x+8^2y+8z+m$.
5. $10(x+3)+x$ which $=11x+30$.

6. $10x+\frac{24}{x}$.

7. Let x and y be the digits, the order being $x\,y$ in the first number, and $y\,x$ in the second. The algebraical expression for the first number must, therefore, be . . . . ., and the expression for the second must be . . . . .; and since the ratio of the numbers is 3 to 8, we have the equation

. . . . . = ...

Also, since the sum of the digits is 9, we have the equation

. . . . . = ...

From these two equations we get $x=2$, and $y=7$. The numbers are, therefore, 27 and 72.

8. Let x denote the first digit to the right of the decimal point, and y the next. Then the algebraical expression for the decimal fraction will be . . . . . And, since by subtracting ·18 from this fraction the digits exchange places, we get the equation

. . . . . = ...

Again, since $\frac{7}{4}$ of the fraction which is expressed by the reversed digits is the original fraction, we have the equation

. . . . . = ...

From these two equations we get $x=4$, $y=2$. Hence the decimal fraction is ·42.

Another Solution.

Let x denote the original decimal fraction, and let y denote the decimal fraction formed by the digits of the first when reversed. Then, since by subtracting ·18 from the first fraction we get the second, we have the equation

. . . . . = ...

And, since $\frac{7}{4}$ of the second fraction is equal to the first, we have also the equation

$$\ldots\ldots = \ldots\ldots$$

From these two equations we get $x = \frac{21}{50}$, $y = \frac{6}{25}$; that is, $x = \cdot 42$, $y = \cdot 24$.

L.

1. $\frac{x}{9}$.
2. $\frac{10^2x+10y+z}{999}$.
3. $\frac{10x+y-x}{90}$, which $= \frac{9x+y}{90}$.
4. $\cdot 2\dot{3}$.

5. He purchased the estate for x pounds, suppose, and let it at once at a yearly rent of .... pounds. He left y pounds, suppose, of the purchase-money unpaid; and the yearly interest on this part, at 4 per cent. per annum, being, we are told, equal to $\frac{2}{3}$ of the rent, we have the equation

$$\ldots\ldots = \ldots\ldots$$

from which we get $y = \frac{x}{2}$. He afterwards paid off ..... pounds of his debt, and on the remaining .... pounds still due he paid a yearly interest of .... pounds. This being deducted from his yearly rent of .... pounds, leaves him, we are told, a yearly income of 180 pounds. This gives us the equation

$$\ldots\ldots = 180,$$

from which we get $x = 10{,}000$.

The purchase-money of the estate was, therefore, 10,000*l.*

APPENDIX

ON

THE RESOLUTION OF ALGEBRAICAL EXPRESSIONS INTO FACTORS.

INTRODUCTORY EXERCISES.

Expressions which may be resolved into their elementary factors by inspection.

1. Apply the formula $a^2-b^2=(a-b)(a+b)$.

(1) x^2-a^2
(2) x^4-y^4
(3) x^4-1
(4) x^2-4
(5) x^2-25
(6) $1-x^2$
(7) $4x^2-9$
(8) $1-4x^2$
(9) $9x^2-4a^2$
(10) $16-a^4x^4$
(11) $x^2-a^2y^2$
(12) $4a^2-9x^2y^2$
(13) $9-25a^2x^2$
(14) $16a^2b^2-a^2$
(15) $16a^2-9$
(16) $12x^2-3y^2$
(17) $9a^2b^2-4x^2y^2$
(18) $2a^2x^2-18x^2y^2$
(19) $28a^2b^2-63c^2$
(20) $6x^2-24y^2z^2$
(21) $(a-b)^2-c^2$
(22) $x^2-(y+z)^2$
(23) $(a+b)^2-(c-d)^2$
(24) $4\,(a+b)^2-9c^2$
(25) $x^2(a+b)^2-c^2y^2$
(26) $ax^2-ab^2c^2$
(27) $ax^2-a(y-z)^2$
(28) $x^3-x\,(y+z)^2$
(29) $x-x^3$
(30) $4a^2x^3-9a^2x$

2. Apply the formula $a^2\pm 2ab+b^2=(a\pm b)^2$.

(1) $x^2+2ax+a^2$; $x^2-2ax+a^2$
(2) x^2-2x+1; x^2+2x+1
(3) x^2+4x+4; x^2-4x+4
(4) $x^2+4xy+4y^2$; $x^2-4xy+4y^2$
(5) $x^2-6xy+9y^2$; $x^2+6xy+9y^2$
(6) $x^3+4x^2y+4xy^2$;
$x^3-4x^2y+4xy^2$
(7) $4x^2-12xy+9y^2$;
$4x^2+12xy+9y^2$
(8) $x^4+2x^2y^2+y^4$
(9) $x^4-2x^2y^2+y^4$
(10) $4x^4-12x^2y^2+9y^4$
(11) $a^5-2a^3b^2+ab^4$
(12) $3x^4-6a^2x^2+3a^4$
(13) $2x^3-8x^2y+8xy^2$
(14) $a^3xy+2a^2x^2y^2+ax^3y^3$

3. Apply the three formulæ $a^2-b^2=(a-b)(a+b)$,
$a^3-b^3=(a-b)(a^2+ab+b^2)$,
and $(a^3+b^3)=(a+b)(a^2-ab+b^2)$.

(1) x^3-y^3
(2) x^3+y^3
(3) x^6-y^6
(4) x^3-1
(5) $1+x^3$
(6) x^3+8
(7) x^3+8a^3
(8) $8a^3-27x^3$
(9) $a^3-x^3y^3$
(10) ax^4-a^4x
(11) $8x^6-8a^6$
(12) $-x^3+y^3$
(13) $-x^3-y^3$
(14) a^3+1
(15) x^3-64
(16) a^8-x^8
(17) $1-a^3y^3$
(18) $8+a^3x^3$
(19) $8a^3x^2-8x^5y^3$
(20) x^5-x^2

CHAPTER I.

To find the factors of any algebraical expression which can be put into the form Mq+N.

If some of the terms of an algebraical expression contain any factor q *of the first power only*, the expression can be put into the form $Mq+N$, in which M denotes the sum of the coefficients of q, and N denotes the sum of the terms which do not contain q.

When the expression is thus arranged, any factor common to M *and* N *will be a factor of the whole expression* Mq+N; *and if* M *and* N *have no common factor, the expression* Mq+N *cannot be resolved into factors.*

As a simple application of this theorem, let it be required to resolve $8x^2+12ax+10bx+15ab$ into its elementary factors.

We observe that the factor a enters into two of the terms, while the higher powers of a do not appear in the expression. We therefore collect the coefficients of a and put the given expression into the form $a(12x+15b)+(8x^2+10bx)$. Now, by the theorem, the two brackets ought to have a common factor, and it is evident, by mere inspection, that this is the case; for the first bracket resolves into $3(4x+5b)$, and the second resolves into $2x(4x+5b)$, the common factor being $4x+5b$. The given expression therefore $=(4x+5b)(2x+3a)$.

As another example, take the expression

$$x^3+(a-b-c)x^2+(bc-ca-ab)x+abc.$$

When we have cleared the expression of brackets, the factor a will appear in several of the terms, while the higher powers of a will not appear in the expression. Collecting, therefore, the coefficients of a, the given expression

$$\begin{aligned} &= a(x^2-bx-cx+bc)+(x^3-bx^2-cx^2+bcx) \\ &= a(x^2-bx-cx+bc)+x(x^2-bx-cx+bc) \\ &= (a+x)(x^2-bx-cx+bc). \end{aligned}$$

In a similar manner, by collecting the coefficients of b, the second bracket may be put into the form $b(c-x)+(x^2-cx)$, which evidently $= b(c-x)+x(x-c) = b(c-x)-x(c-x)$ $= (b-x)(c-x) = (x-b)(x-c)$. Hence the given expression $= (x+a)(x-b)(x-c)$.

As another example, let it be required to find the elementary factors of the expression

$$a^2x^3-a^3x^2y-a^2xy+a^3y^2-ax^2yz+x^3z-xyz+ay^2z.$$

We first notice here that the factor z, *of the first degree only*, appears in some of the terms. We therefore collect the coefficients of z, and put the expression into the form

$$z(x^3-ax^2y-xy+ay^2)+(a^2x^3-a^3x^2y-a^2xy+a^3y^2).$$

It is evident that a^2 is a factor of the second bracket, so that the whole expression may be put into the form

$$z(x^3-ax^2y-xy+ay^2)+a^2(x^3-ax^2y-xy+ay^2),$$

which now evidently $= (z+a^2)(x^3-ax^2y-xy+ay^2)$.

In the second bracket of this last expression we notice that a appears as a factor *of the first degree only* in two terms. We therefore collect the coefficients of a, and put the expression in the second bracket into the form $a(y^2-x^2y)+(x^3-xy)$, which, when we take outside the brackets the monomial factors y and x, becomes $ay(y-x^2)$ $+x(x^2-y)$, and this again $= ay(y-x^2)-x(y-x^2)$ $= (ay-x)(y-x^2)$. Hence, $a^2x^3-a^3x^2y-a^2xy+a^3y^2-ax^2yz$ $+x^3z-xyz+ay^2z$, when resolved into its elementary factors, becomes $(z+a^2)(ay-x)(y-x^2)$.

Next, take the expression $x^2-3ax+4bx+12ab$. Collecting the coefficients of b, this expression may be put into the form $b(4x+12a)+(x^2-3ax)$. Now, by our theorem, if the given expression can be resolved into factors, the expressions in the two brackets must have a common factor. But they have not; for the expression in the first bracket $= 4(x+3a)$, while that in the second bracket $= x(x-3a)$. The given expression cannot therefore be resolved into factors.

Again; take the expression $2x^3+6x^2y^2+8x^2z+xy+3y^3+4yz$. We notice that z, as a factor *of the first degree only*, enters into two of the terms. We therefore collect the coefficients of z, and put the expression into the form

$$z(8x^2+4y)+(2x^3+6x^2y^2+xy+3y^3).$$

Now, by the theorem, if this expression can be resolved into elementary factors, one of these factors will be (or be a factor of) the G.C.M. of the expressions in the brackets. The G.C.M. of these expressions is readily found to be $2x^2+y$, for by inspection we get $8x^2+4y = 4(2x^2+y)$, and then by simple division we get $2x^3+6x^2y^2+xy+3y^3 = (2x^2+y)(x+3y^2)$. This G.C.M., namely, $2x^2+y$, is therefore a factor of the whole expression to be resolved, which now becomes $(2x^2+y)(4z+x+3y^2)$.

We may in a similar manner resolve into factors expressions which may be put into the form Mq^n+N, in which M denotes the sum of the coefficients of q^n, and N denotes the sum of the terms which do not contain q^n, *no other power of* q *appearing in the expression.*

Take, for example, the expression $12a^2b^3+8a^2x-3b^3x^2-2x^3$. We notice that a^2 is a factor of two terms, while the other powers of a do not appear in the expression. We therefore collect the coefficients of a^2, and put the expression into the form $a^2(12b^3+8x)-(3b^3x^2+2x^3)$. It is easily seen that the G.C.M. of the expressions in the two brackets is $3b^3+2x$, for $12b^3+8x = 4(3b^3+2x)$ and $3b^3x^2+2x^3 = x^2(3b^3+2x)$; and therefore the whole expression $12a^2b^3+8a^2x-3b^3x^2-2x^3 = (3b^3+2x)(4a^2-x^2)$; and resolving the second bracket by inspection, this becomes $(3b^3+2x)(2a+x)(2a-x)$.

Or, we may collect the coefficients of b^3, since no other power of b appears in the expression, and put the expression into the form $b^3(12a^2-3x^2)+(8a^2x-2x^3)$. The G.C.M. of the expressions in the two brackets is easily seen to be $4a^2-x^2$; this, therefore, is a factor of the whole expression, which may now be put into the form $(4a^2-x^2)(3b^3+2x)$, which again resolves into $(2a-x)(2a+x)(3b^3+2x)$.

1. Resolve into their elementary factors the following expressions, all of which may be put either into the form $Mq+N$, or into the form Mq^n+N.

(1) $ac+bd+bc+ad$.
(2) $2x^2+4ax+3bx+6ab$.
(3) $x^3y-x^2z-xy^2+yz$.
(4) $bc+ab-ac-b^2$.
(5) $2x^3y+2bx^4-bx^3y+4abx^2y-x^2y^2+4axy^2-2abxy^2-2ay^3$.
(6) $abxy+b^2y^2+acx-c^2$.
(7) $x^2z^2+ax^2-a^2z^2-a^3$.
(8) $x^5-a^3x^2-b^2x^3+a^3b^2$.
(9) $x^3+(a-b-c)x^2-(ab-bc+ca)x+abc$.

2. Show that $x^3+xz+xy^2+yz$ cannot be resolved into factors.

3. Find the values of x which satisfy the following equations:—

(1) $2x^2-ax-4bx+2ab=0$.
(2) $x^2-ax-bx+ab=0$.
(3) $x^2-(a+2b+c)x+ac+(a+c)b+b^2=0$.
(4) $x^3-b^2x^2-a^2x+a^2b^2=0$.

4. Find the least common multiple of the three expressions $2x^2-3ax+4bx-6ab$, $2x^2+3ax-4bx-6ab$, and $x^2+2bx+3ax+6ab$.

CHAPTER II.

To find the factors of any algebraical expression which may be put into the form Mq^2+Nq+P.

If some of the terms of any algebraical expression contain any factor q, and others the factor q^2, the other powers of q not appearing in the expression, the expression may be put into the form Mq^2+Nq+P, in which M denotes the sum of the coefficients of q^2, N denotes the sum of the coefficients of q, and P the sum of the terms which do not contain q.

When the expression is thus arranged, the factors of Mq^2+Nq+P can always be found by solving the quadratic equation $Mq^2+Nq+P=O$; for if α and β be the roots of this equation, so that $\alpha=\frac{-N+\sqrt{N^2-4M\cdot P}}{2M}$, and $\beta=\frac{-N-\sqrt{N^2-4M\cdot P}}{2M}$, we shall have Mq^2+Nq+P $=M(q-\alpha)(q-\beta)$.

Take for example the expression $20x^3-15x^2y-8xy+6y^2$. Here we notice that no power of y higher than the second appears in the expression; we therefore arrange the expression in descending powers of y and put it into the form $6y^2-(15x^2+8x)y+20x^3$. Solving as a quadratic *with regard to* y, we find that this expression vanishes when $y=\frac{5x^2}{2}$ and when $y=\frac{4x}{3}$. Hence the given expression $=6\left(y-\frac{5x^2}{2}\right)\left(y-\frac{4x}{3}\right)$; and this, by simplification, becomes $(2y-5x^2)(3y-4x)$.

This process, however, though perfect in theory, is usually inconvenient in practice. As a general rule, it will be better to find the factors of the expression Mq^2+Nq+P by a tentative process, as follows :—

Assume the factors to be

$$\begin{gathered} Aq+B \\ A_1q+B_1. \end{gathered}$$

Since the product of these factors is to be Mq^2+Nq+P, it is evident that they must satisfy the following three conditions:—

(1) $Aq\times A_1q = Mq^2$; that is to say, the product of the first terms of the factors must equal first term of the given expression.

(2) $B\times B_1 = P$; that is to say, the product of the second terms of the factors must equal third term of the given expression.

(3) $Aq\times B_1+A_1q\times B = Nq$; that is to say, the sum of the products obtained by multiplying the terms diagonally opposite must equal the middle term of the given expression.

We can generally find by mere inspection values of A and A_1 to satisfy the first condition, and values of B and B_1 to satisfy the second. These being found, we must apply the third condition as a test of the correctness of these values. If the third condition is satisfied *in all but the sign of the middle term*; that is to say, if the sum of the products obtained by the cross multiplication $= -Nq$, when it ought to be equal to $+Nq$, it is evident that we shall have the *true* factors by changing the signs of the second terms of the *trial* factors; or, if it should be more convenient, by changing the signs of their first terms. That is to say, if the *trial* factors be

$$\begin{gathered} Aq+B \\ A_1q+B_1 \end{gathered}$$

the *true* factors will be

$$\begin{matrix} Aq-B \\ A_1q-B_1 \end{matrix} \text{ or } \begin{matrix} B-Aq \\ B_1-A_1q, \end{matrix}$$

whichever we please; for the product of the one pair is evidently equal to the product of the other pair.

Take for example the quadratic expression $5x^2+13x-6$. Try as factors

$$\begin{array}{c} 5x+2 \\ x-3. \end{array}$$

The third condition is that $5x \times (-3) + 2 \times x$ must be equal to $+13x$; and this condition would be satisfied if the middle term were $-13x$ instead of $+13x$. Consequently, the *true* factors are

$$\begin{array}{c} 5x-2 \\ x+3. \end{array}$$

As another example, take the expression $2a^2 - b^3c - ab^2 - bc^3 - ac^2 + 2b^2c^2 + 4abc$. Arranging this expression in descending powers of a, it becomes $2a^2 - a(b^2 + c^2 - 4bc) - (b^3c + bc^3 - 2b^2c^2)$. We find by inspection that bc is a factor of the third term, which therefore $= bc(b^2 + c^2 - 2bc) = bc(b-c)^2$. The expression to be resolved is therefore

$$2a^2 - a(b^2 + c^2 - 4bc) - bc(b-c)^2.$$

We find by mere inspection of the first and third terms of this expression that the *trial* factors

$$\begin{array}{c} 2a + (b-c)^2 \\ a - bc \end{array}$$

will satisfy the first two conditions; and we find by trial that the third condition would be satisfied if only the sign of the middle term in the given expression were positive instead of negative. Consequently the *true* factors cf the expression are

$$\begin{array}{c} 2a - (b-c)^2 \\ a + bc. \end{array}$$

These will satisfy the *three* conditions.

As another example, take the expression $bc^2 - ac^2 - b^2c + ab^2 - a^2b + a^2c$. Arranging this expression in descending powers of c, it becomes $(b-a)c^2 - (b^2 - a^2)c + ab(b-a)$. It is now evident that $b-a$ is a factor of every term, and consequently of the whole expression, which may accordingly be put into the form $(b-a)(c^2 - bc - ac + ab)$. By the theorem of the preceding chapter, the expression in the second bracket is readily resolved into the form

$(b-c)(a-c)$. The given expression therefore $=(b-a)(b-c)(a-c)$, or more symmetrically, by changing the signs of two of its factors $(a-b)(b-c)(c-a)$.

As another example, take the expression $x^4+x^3(y+z)+x^2(y^2+yz+z^2)+xyz(y+z)+y^2z^2$. When we clear the expression of brackets, we notice that y and y^2 appear as factors of several terms, while the higher powers of y do not enter into the expression. We therefore arrange the expression according to descending powers of y, and put it into the form

$$y^2(x^2+xz+z^2)+y(x^3+x^2z+xz^2)+(x^4+x^3z+x^2z^2).$$

It is now evident that x^2+xz+z^2 is a factor of every term, for the expression in the second bracket resolves into $x(x^2+xz+z^2)$, and the expression in the third bracket resolves into $x^2(x^2+xz+z^2)$; consequently, x^2+xz+z^2 is a factor of the whole expression, which may now be put into the form $(x^2+xz+z^2)(y^2+xy+x^2)$.

As another example, take the expression $2x^2-7xy-3ax+3y^2+4ay+a^2$. As both the first and second power of each quantity enter into the expression, it is immaterial whether we arrange it according to the powers of x, y, or a; we will take the last, and put it into the form $a^2+a(4y-3x)+(2x^2-7xy+3y^2)$.

The factors of the third term of this expression are easily found to be $2x-y$ and $x-3y$; so that the first and second conditions will be satisfied by the trial factors

$$\begin{aligned}&a+(2x-y)\\&a+(x-3y).\end{aligned}$$

The third condition will also be satisfied *in all but the sign of the middle term of the product*, which we shall find to be $a(3x-4y)$ instead of $a(4y-3x)$. Hence the true factors are

$$\begin{aligned}&a-(2x-y)\\&a-(x-3y),\end{aligned}$$

and the given expression becomes $(a-2x+y)(a-x+3y)$.

In the choice of trial factors it should be observed that no factor is admissible which is divisible by any number or quantity which does not also divide the expression to be resolved. For example, $2x+4$ cannot be a factor of the expression $6x^2+17x+12$, because $2x+4$ is divisible by 2, while $6x^2+17x+12$ is not.

1. Resolve the following expressions into their elementary factors :—

(1) $x^2+8x+15$; $x^2-8x+15$.
(2) $2x^2+7x+6$; $2x^2-7x+6$.
(3) $x^2-2x-15$; $x^2+2x-15$.
(4) $10x^2+29x-21$; $10x^2-29x-21$.
(5) $4x^2-49$.
(6) $x^2-x-110$; $x^2+x-110$.
(7) $x^2+ax-2a^2$; $x^2-ax-2a^2$.
(8) $4x^2-8xy+3y^2$; $4x^2+8xy+3y^2$.
(9) $9x^2-25y^2$.
(10) $x^3-ax^2-ax+a^2$; $x^3+ax^2+ax+a^2$.
(11) $6x^3y+x^2(7y^2-12a)-x(3y^3+14ay)+6ay^2$.
(12) $a^2c+ab^2-b^2c-ac^2-a^2b+bc^2$.
(13) $a^4+a^3(b+c)+a^2(b^2+bc+c^2)+abc(b+c)+b^2c^2$.
(14) $2a^2-2b^2-c^2+3ab+3bc-ac$.
(15) $2x^4y^2+5x^3y^3+x^2(3y^4-2z^2)-5xyz^2-3y^2z^2$.
(16) $x^2-4y^2+12yz-9z^2$.

2. Find what values of x satisfy the following equations :—

(1) $x^2-5x+6=0$; $x^2+5x+6=0$.
(2) $x^2+x-12=0$; $x^2-x-12=0$.
(3) $12x^2-37x+21=0$; $12x^2+37x+21=0$.
(4) $6x^2+17x-3=0$; $6x^2-17x-3=0$.
(5) $x^2+ax-6a^2=0$; $x^2-ax-6a^2=0$.
(6) $x^3-ax^2-ax+a^2=0$.
(7) $24x^4+4bx^3-2x^2(3a^2+4b^2)-a^2bx+2a^2b^2=0$.

3. Find the greatest common measure, and the least common multiple of $a^2c+ab^2-b^2c-ac^2-a^2b+bc^2$, and $a^2c^2-3abc^2+2b^2c^2-a^4+3a^2b-2a^2b^2$.

CHAPTER III.

To find the factors of any algebraical expression which can be put into the form $\mathrm{M}q^3+\mathrm{N}q^2+\mathrm{P}q+\mathrm{Q}$.

If some of the terms of an algebraical expression contain any factor q, others the factor q^2, and others the factor q^3, while the higher powers of q do not enter into the expression, the expression may be put into the form Mq^3+Nq^2+Pq+Q, in which M denotes the sum of the coefficients of q^3, N denotes the sum of the coefficients of q^2, P denotes the sum of the coefficients of q, and Q denotes the sum of the terms which do not contain q.

The expression being thus arranged, assume the factors to be

$$\begin{array}{r} Aq^2+Bq+C \\ B_1q+C_1. \end{array}$$

If these are the true factors, since their product is to be the given expression, it is evident that they must satisfy the following four conditions:—

(1) $A\times B_1=M$; that is to say, the product of the first terms of the factors must equal the first term of the given expression; or, which is the same thing, the product of the coefficient of q^2 in one factor, and the coefficient of q in the other, must equal the coefficient of q^3 in the given expression.

(2) $C\times C_1=Q$; that is to say, the product of the last terms of the factors must equal the last term of the given expression.

(3) $B\times C_1+B_1\times C=P$; that is to say, the coefficient of q in the product of the factors must equal the coefficient of q in the given expression. Observe here that the multiplication is performed diagonally in both cases.

(4) $A\times C_1+B\times B_1=N$; that is to say, the coefficient of q^2 in the product of the factors must equal the coefficient

of q^2 in the given expression. Observe here that the first multiplication is performed diagonally, and the second perpendicularly.

We can generally find by inspection values of A and B_1 to satisfy the first of these conditions, and values of C and C_1 to satisfy the second. These being found, the third condition gives us a simple equation, from which we readily obtain B. The fourth condition is a final test of the correctness of these values.

Take for example, the expression $14x^3+25x^2+41x+10$. The trial factors

$$\begin{array}{r} 7x^2+mx+5 \\ 2x+2 \end{array}$$

would satisfy the first two conditions, since 7 and 2 are complementary factors of 14, and 5 and 2 are complementary factors of 10; but it is evident, without proceeding further, that these are not the true factors, since both terms of the lower trial factor are divisible by 2, while *all* the terms of the expression to be resolved do not contain the same factor, for 2 is neither a factor of 25 nor of 41. Next try as factors

$$\begin{array}{r} 7x^2+mx+2 \\ 2x+5. \end{array}$$

The third condition gives us the equation $5m+4=41$, from which we get $m=\frac{37}{5}$. Since m must be an integer (in consequence of the coefficients of the powers of x in the given expression being integers), we need not apply the test supplied by the fourth condition; *without this test we know that these cannot be the true factors.* Let us next try the factors

$$\begin{array}{r} 2x^2+mx+5 \\ 7x+2. \end{array}$$

The third condition gives us the equation $2m+35=41$, from which we get $m=3$; and since this value of m also satisfies the fourth condition, namely, $4+7m=25$; we know that it is correct and that we have the true factors; that is to say, we get

$$14x^3+25x^2+41x+10 = (2x^2+3x+5)(7x+2).$$

We might of course obtain the quadratic factor by actual division; but the method which we have adopted, besides being more expeditious, is a particular instance of a general and systematic method, to be given hereafter, which will enable us to find the factors in cases to which the process of division would be utterly inapplicable.

Unsuccessful trials may frequently be avoided by attention to the following facts:—

(*a*) If all the terms of the expression to be resolved are positive, all the terms of each factor must also be positive.

(*b*) If the terms of the expression to be resolved are alternately positive and negative, the terms of each factor must also be alternately positive and negative.

(*c*) If all the terms of the expression to be resolved are integral, none of the terms of any factor must be fractional.

(*d*) If there is no factor common to all the terms of the expression to be resolved, there must be no factor common to all the terms of any of its factors.

As the method explained in this chapter is included in the more general method explained in the next, one more example will suffice. Let it be required to resolve the expression

$$10x^4-x^3(15y+4z)-x^2(40y^2-6yz)+x(60y^3+16y^2z)-24y^3z$$

into its elementary factors.

When this expression is cleared of brackets and written out in full, the first thing that we notice is that z, *as a factor of the first power only*, is common to several of the terms, the higher powers of z not appearing in the expression. Employing, therefore, the method of Chapter I., we collect the coefficients of z and put the expression into the form

$$z(-4x^3+6x^2y+16xy^2-24y^3)+(10x^4-15x^3y-40x^2y^2 \\ +60xy^3).$$

By the theorem in Chapter I., if this expression can be resolved into factors, the expressions in the brackets must have a common factor; and it is easily seen that this is the case, for the expression in the first bracket may be resolved into the form $-2(2x^3-3x^2y-8xy^2+12y^3)$, and the expression in the second bracket may be resolved into the form $5x(2x^3-3x^2y-8xy^2+12y^3)$, the common factor being $2x^3-3x^2y-8xy^2+12y^3$. This common factor is therefore a factor of the whole expression, which may now be put into the form $(5x-2z)(2x^3-3x^2y-8xy^2+12y^3)$.

Since, in the expression in the second bracket, the powers of x regularly decrease, while those of y regularly increase, it will simplify matters if we assume that this is also the case in its factors. Suppose, therefore, we try as its factors

$$\begin{array}{r} 2x^2+mxy+4y^2 \\ x+3y, \end{array} \quad \begin{array}{c} \text{or, writing only numerical} \\ \text{coefficients,} \end{array} \quad \begin{array}{r} 2+m+4 \\ 1+3. \end{array}$$

The third condition is $3m+4=-8$, from which we get $m=-4$. Without applying the fourth condition, it is now evident that these are not the true factors, since every term of the upper trial factor is divisible by 2, while 2 does *not* divide every term of the expression to be resolved. Let us next try as factors

$$\begin{array}{r} 2x^2+mxy+3y^2 \\ x+4y, \end{array} \quad \text{or, writing coefficients only,} \quad \begin{array}{r} 2+m+3 \\ 1+4. \end{array}$$

The third condition is $4m+3=-8$. Since this would give a fractional value to m, we know, without proceeding further, that these are not the true factors either. Next, let us try as factors

$$\begin{array}{r} 2x^2+mxy\quad 6y^2 \\ x+2y, \end{array} \quad \text{or, writing coefficients only,} \quad \begin{array}{r} 2+m+6 \\ 1+2. \end{array}$$

The third condition is $2m+6=-8$, from which we get $m=-7$; and since this value of m satisfies also the fourth condition, namely, $4+m=-3$, we know that these are the true factors. Therefore,

$$2x^3-3x^2y-8xy^2+12y^3 = (2x^2-7xy+6y^2)(x+2y);$$

and the whole expression is now resolved into the form $(5x-2z)(x+2y)(2x^2-7xy+6y^2)$. The third bracket being easily resolved, by the method of the preceding chapter, into the form $(2x-3y)(x-2y)$, the whole expression is finally resolved into the form

$$(5x-2z)(x+2y)(2x-3y)(x-2y).$$

1. Resolve the following expressions into their elementary factors:—

(1) $x^3+9x^2+16x+4$.
(2) $x^3-9x^2+16x-4$.
(3) $4x^3+16x^2+19x+6$.
(4) $x^3-7x^2+16x-12$.
(5) $x^3-7x^2y+16xy^2-12y^3$.
(6) $2x^3+5x^2y+4xy^2+y^3$.
(7) $x^3-9x^2y+16xy^2-4y^3$.
(8) $x^4+x^3(10y+z)+x^2(25y^2+9yz)+x(20y^3+16y^2z)$
$+4(y^4+y^3z)$.
(9) $x^4+2ax^3+(a^2+a)x^2+2a^2x+a^3$.

2. Find the values of x which satisfy the following equations:—

(1) $x^3-6ax^2+11a^2x-6a^3=0$.
(2) $x^3+6ax^2+11a^2x+6a^3=0$.
(3) $6x^3-11ax^2+6a^2x-a^3=0$.

CHAPTER IV.

To find the quadratic factors of any expression which may be put into the form $Mq^4+Nq^3+Pq^2+Qq+R$; in which M denotes the coefficient, or the sum of the coefficients of q^4, N the sum of the coefficients of q^3, and so on. (See the preceding chapter.)

Assume the factors to be

$$Aq+Bq+C$$
$$A_1q^2+B_1q+C_1.$$

These must satisfy the following five conditions:

(1) $A\times A_1=M$; or, which is the same thing, the product of the first terms of the factors must equal the first term of the expression to be resolved.

(2) $C\times C_1=R$; that is to say, the product of the last terms of the factors must equal the last term of the expression to be resolved.

(3) $(A\times B_1)+(A_1\times B)=N$; that is to say, the coefficient of q^3 in the product of the factors must equal the coefficient of q^3 in the expression to be resolved. Observe here that the multiplication is performed diagonally.

(4) $(B\times C_1)+(B_1\times C)=Q$; that is to say, the coefficient of q in the product of the factors must equal the coefficient of q in the expression to be resolved. As in the preceding condition, observe that the multiplication is performed diagonally.

(5) $(A\times C_1)+(A_1\times C)+(B\times B_1)=P$; that is to say, the coefficient of q^2 in the product of the factors must equal the coefficient of q^2 in the expression to be resolved. Observe here that the multiplication in the first two brackets is performed diagonally, and in the third, perpendicularly.

We can generally find by inspection values of A and A_1; to satisfy the first condition, and values of C and C_1 to satisfy the second. These being found, the third and fourth conditions give us two simultaneous equations from which values of B and B_1 may be obtained. The fifth condition will be a final test of the correctness of these values.

Unsuccessful trials may often be avoided by attention to the facts mentioned in the preceding chapter, and marked (*a*), (*b*), (*c*), (*d*).

As an example of the resolution of a biquadratic expression into its quadratic factors, take $x^4+4x^3+8x^2+8x+3$.

Here there is no hesitation; if any rational quadratic factors exist they *must* be of the form,

$$1x^2+mx+3$$
$$1x^2+nx+1,$$

in which m and n denote some integers at present unknown. The third and fourth conditions are

$$m+n=4$$
$$\text{and } m+3n=8,$$

from which we get $m=2$, $n=2$. Substituting these values of m and n in the fifth condition, we find that they will satisfy this condition also. Consequently,

$$x^4+4x^3+8x^2+8x+3=(x^2+2x+3)(x^2+2x+1).$$

Next, let it be required to find the quadratic factors of the expression

$$2x^4-13x^3y+34x^2y^2-46xy^3+35y^4.$$

Since the terms of this expression are alternately positive and negative, the terms of each factor must also be alternately positive and negative. And since the powers of x regularly descend, while the powers of y regularly ascend, we may assume that this is also the case in each factor. Let us first try as factors

$$2x^2+mxy+7y^2$$
$$1x^2+nxy+5y^2,$$

or, writing numerical coefficients only,

$$2+m+7$$
$$1+n+5.$$

The third and fourth conditions are

$$2n+m=-13$$
$$7n+5m=-46,$$

from which we get $n=-\frac{19}{3}$. Without proceeding further, we now know that these are not the true factors, for n and m must both be integers.

Let us next try as factors, writing the numerical coefficients only,

$$\begin{gathered}2+m+5\\1+n+7.\end{gathered}$$

The third and fourth conditions are now

$$\begin{aligned}2n+m&=-13\\5n+7m&=-46,\end{aligned}$$

and from these two equations we get $n=-5$, $m=-3$.

Since these values of m and n also satisfy the fifth condition, we know that these are the true factors. The expression $2x^4-13x^3y+34x^2y^2-46xy^3+35y^4$ is thus resolved into the form $(2x^2-3xy+5y^2)(x^2-5xy+7y^2)$.

Next take the expression $x^4+x^2y^2+y^4$. If we put zero for the numerical coefficients of the missing powers of x and y, this expression may be written $1x^4+0x^3y+1x^2y^2+0xy^3+1y^4$. Assume the quadratic factors to be

$$\begin{gathered}1x^2+mxy+1y^2\\1x^2+nxy+1y^2\end{gathered}$$

or, writing the numerical coefficients only,

$$\begin{gathered}1+m+1\\1+n+1\end{gathered}$$

From either the third or fourth condition we get $m+n=0$, and therefore $n=-m$. Substituting this value of n in the fifth condition $2+mn=1$, we get $m^2=1$ and therefore $m=\pm 1$; and since $n=-m$, either value of m will give us

$$x^4+x^2y^2+y^4=(x^2+xy+y^2)(x^2-xy+y^2).$$

Next, take the expression $a^4-2a^2b^2+b^4+4abc^2-c^4$. Arranging the expression in descending powers of a, and putting zero for the coefficient of the missing third power of a, the expression will become

$$1a^4+0a^3-(2b^2)a^2+(4bc^2)a+(b^4-c^4).$$

Assume the factors to be

$$1a^2+ma+(b^2+c^2)$$
$$1a^2+na+(b^2-c^2)$$

which evidently satisfy the first two conditions, since $1\times 1=1$, and $(b^2+c^2)(b^2-c^2)=b^4-c^4$. The third condition is $m+n=0$, from which we get $n=-m$. Substituting this value of n in the fourth condition

$$m(b^2-c^2)+n(b^2+c^2)=4bc^2,$$

we get $m=-2b$, and therefore $n=+2b$. Finally, since these values of m and n satisfy the fifth condition, namely, $1(b^2-c^2)+1(b^2+c^2)+mn=-2b^2$, we know that we have the true factors; that is to say, we get

$$a^4-2a^2b^2+b^4+4abc^2-c^4$$
$$=(a^2-2ab+b^2+c^2)(a^2+2ab+b^2-c^2).$$

As another example take the expression

$$a^4+8b^2x^2(a^2-2)+16b^4x^4.$$

Arranging this expression in descending powers of x, and putting zero for the coefficient of the missing powers, it will take the form

$$(16b^4)x^4+0x^3+(8a^2b^2-16b^2)x^2+0x+a^4.$$

The trial factors

$$(4b^2)x^2+(m)x+a^2$$
$$(4b^2)x^2+(n)x+a^2$$

will evidently satisfy the first two conditions. Either the third or fourth condition will give us $m=-n$. Substituting this value of m in the fifth condition, we get $n=\pm 4b$, and therefore $m=\mp 4b$, the upper signs going together and the lower signs going together. Hence, whether we take the upper or the lower signs, the given expression resolves into the form

$$(4b^2x^2-4bx+a^2)(4b^2x^2+4bx+a^2).$$

The following example will illustrate the methods of Chapters I. and II., as well as the method of the present chapter.

Resolve into its elementary factors the expression

$$8x^6+12x^5y-(16+2a^2)x^4y^2-(12+3a^2)x^3y^3 \\ +(8+4a^2)x^2y^4+3a^2xy^5-2a^2y^6.$$

Now, the first thing to be noticed in this expression, when it is cleared of brackets and written out in full, is that there is one quantity *of one power only*, namely, the quantity a^2, which enters as a factor into several of the terms. This leads us to suspect (see Chapter I.) that the expression may be resolved into the form $(Aa^2+B)C$, in which A, B, and C denote some algebraical expressions, at present unknown, any of which may involve x or y, but not a. We accordingly collect the coefficients of a^2 and put the expression into the form

$$a^2(-2x^4y^2-3x^3y^3+4x^2y^4+3xy^5-2y^6) \\ +(8x^6+12x^5y-16x^4y^2-12x^3y^3+8x^2y^4).$$

By the theorem of Chapter I., the expressions in the two brackets ought to have a common factor, and it is evident by mere inspection that this is the case; for the first bracket resolves into the form

$$-y^2(2x^4+3x^3y-4x^2y^2-3xy^3+2y^4),$$

and the second bracket resolves into the form

$$4x^2(2x^4+3x^3y-4x^2y^2-3xy^3+2y^4),$$

the common factor being

$$2x^4+3x^3y-4x^2y^2-3xy^3+2y^4.$$

This common factor is therefore a factor of the whole expression, which may now be put into the form

$$(2x^4+3x^3y-4x^2y^2-3xy^3+2y^4)(4x^2-a^2y^2);$$

and, resolving the second bracket by inspection, this becomes

$$(2x^4+3x^3y-4x^2y^2-3xy^3+2y^4)(2x-ay)(2x+ay).$$

Since in the first bracket the powers of x regularly descend, while the powers of y regularly ascend, it will simplify matters if we assume that this is also the case in each of its factors. The trial factors

$$\begin{array}{l}2x^2+mxy-2y^2\\1x^2+nxy-1y^2\end{array}$$

evidently satisfy the first two conditions. The third and fourth conditions (which in the present case happen to coincide) will both give us $m+2n=3$; and the fifth condition when simplified becomes $mn=0$. From the last equation either $m=0$ or $n=0$; but m cannot $=0$, for then the preceding equation would give us $n=\frac{3}{2}$, which is an inadmissible value, since we do not want fractional coefficients. Consequently, $n=0$ and $m=3$.

All the five conditions being now satisfied, we know the factors of the expression in the first bracket to be

$$\begin{array}{l}2x^2+3xy-2y^2\\1x^2+0xy-1y^2\end{array}$$

The upper factor may be resolved into the form

$$(2x-y)(x+2y)$$

by the method given in Chap. II.; and the lower factor resolves by inspection into the form $(x-y)(x+y)$.

Thus the whole expression is resolved into its elementary factors and becomes

$$(2x-y)(x+2y)(x-y)(x+y)(2x-ay)(2x+ay).$$

1. Resolve each of the following expressions into two quadratic factors:—

(1) $x^4+9x^3+20x^2+37x+5$.
(2) $x^4-9x^3+20x^2-37x+5$.
(3) $x^4+9x^3y+20x^2y^2+37xy^3+5y^4$.
(4) $x^4-9x^3y+20x^2y^2-37xy^3+5y^4$.
(5) $4x^4-12x^3+29x^2-30x+25$.
(6) $4x^4+12x^3+29x^2+30x+25$.
(7) $4a^4+12a^3b+29a^2b^2+30ab^3+25b^4$.
(8) $4x^4+4x^3+13x^2-3x-12$.
(9) $4x^4-4x^3+13x^2+3x-12$.
(10) $10x^4-49x^3y+57x^2y^2-32xy^3+14y^4$.
(11) $x^4+a^2x^2+a^4$.

(12) x^4+x^2+1.
(13) x^4-39x^2+25.
(14) $x^4-39x^2y^2+25y^4$.
(15) $4x^4-37a^2x^2+9a^4$.
(16) x^4-3x^2+1.
(17) $a^4-3a^2b^2+b^4$.
(18) $9x^4-49x^2y^2+16y^4$.

2. Resolve the following expressions into their elementary factors:—

(1) $6x^4+29x^3+45x^2+24x+4$.
(2) $6x^4+29x^3y+45x^2y^2+24xy^3+4y^4$.
(3) $6x^4-29x^3y+45x^2y^2-24xy^3+4y^4$.
(4) x^4+x^3+x+1.
(5) $a^4+a^3b+ab^3+b^4$.
(6) $x^4-5a^2x^2+4a^4$.
(7) $a^4+3a^2b^2+4b^4$.
(8) $6x^4+17x^3y+26x^2y^2+19xy^3+4y^4$.
(9) $x^4-4y^4+12y^2z^2-9z^4$.
(10) $x^4+y^4-z^4-2x^2y^2+4xyz^2$.
(11) $x^4-2x^3y+5x^2y^2-4xy^3+4y^4$.
(12) $x^6+x^2y^2(x^2+y^2)(1-z^2)-y^6z^2$.
(13) $x^7-y^7z^3+x^5y^2-x^2y^5z^3+x^3y^4-x^4y^3z^3$.

3. Solve the following equations:—

(1) $x^4-19x^3+95x^2-47x+6=0$.
(2) $x^4-19ax^3+95a^2x^2-47a^3x+6a^4=0$.
(3) $x^4 7x^3+14x^2-13x+7=0$.

GENERAL SUMMARY AND EXTENSION

OF THE

METHODS IN THE PRECEDING CHAPTERS.

FIRST see whether any quantity *of the first power only*, or *of one power only*, is a factor common to some of the terms of the expression to be resolved, the other powers of this quantity not appearing in the expression; if so, the expression may be put into the form $Mq+N$, or Mq^n+N, and resolved into factors by the method of Chapter I.

If the expression cannot be put into this form, put it if possible into the form Mq^2+Nq+P, and resolve as in Chapter II.

If the expression can neither be put into the form $Mq+N$, nor into the form Mq^2+Nq+P, put it if possible into the form Mq^3+Nq^2+Pq+Q, or, failing this, into the form $Mq^4+Nq^3+Pq^2+Qq+R$, and so on; and resolve as in Chapters III. and IV., the general principle of which will appear from the following examples:—

First take the expression

$$6x^{10}+2x^9y+15x^8y^2+18x^7y^3+5x^6y^4+29x^5y^5+16x^4y^6$$
$$+13x^3y^7+16x^2y^8+5xy^9+10y^{10}.$$

Assume the factors to be

$$2x^5+mx^4y+nx^3y^2+px^2y^3+qxy^4+5y^5$$
$$3x^5+m_1x^4y+n_1x^3y^2+p_1x^2y^3+q_1xy^4+2y^5$$

in which m, m_1, n, n_1, &c., are integers at present unknown. It will usually be more convenient to write the numerical coefficients only, thus:—

$$2+m+n+p+q+5$$
$$3+m_1+n_1+p_1+q_1+2.$$

The first two conditions are evidently satisfied, since $2 \times 3 = 6$, and $5 \times 2 = 10$. The 3rd, 5th, 7th, 9th, and 11th conditions are

(3) . . $(2m_1+3m) = 2.$
(5) . . $(2n_1+3n)+mm_1 = 15.$
(7) . . $(2p_1+3p)+(mn_1+m_1n) = 18.$
(9) . . $(2q_1+3q)+(mp_1+m_1p)+nn_1 = 5.$
(11) . . $(2\times 2+3\times 5)+(mq_1+m_1q)+(np_1+n_1p) = 29.$

Observe how in each bracket the multiplication is performed diagonally.

The 4th, 6th, 8th, and 10th conditions are

(4) . . $(5q_1+2q) = 5.$
(6) . . $(5p_1+2p)+qq_1 = 16.$
(8) . . $(5n_1+2n)+(qp_1+q_1p) = 13.$
(10) . . $(5m_1+2m)+(qn_1+q_1n)+pp^1 = 16.$

From any eight of the above nine equations we shall get $m=0$, $m_1=1$, $n=5$, $n_1=0$, $p=3$, $p_1=2$, $q=0$, $q_1=1$; and since these values also satisfy the remaining equation, we know that they are correct, and that we have taken the true factors.

Next, take the expression $x^5+7x^4+20x^3+42x^2+26x+3$. In order that the highest power of x may be even, we shall put the expression into the form

$$0x^6+1x^5+7x^4+20x^3+42x^2+26x+3,$$

and assume the factors to be

$$1x^3+mx^2+nx+1$$
$$0x^3+1x^2+px+3,$$

which, as is evident by inspection, satisfy the first three conditions. From the fourth condition we get $p = 26-3n$. Substituting this value of p in the fifth condition, we get $m = 3n-19$. Substituting again these values of p and m in the sixth condition, we get the equation $3n^2-35n$

$+98 = 0$. Since we know that n must be an *integer*, we need not go through the usual tedious process of solving a quadratic. By Horner's expeditious process, or otherwise, we readily get $n = 7$; and therefore $p = 5$, and $m = 2$. These values of m, n, and p will also be found to satisfy the seventh condition; consequently,

$$x^5 + 7x^4 + 20x^3 + 42x^2 + 26x + 3 = (x^3 + 2x^2 + 7x + 1)(x^2 + 5x + 3).$$

The zero coefficients in this example have been introduced merely to facilitate the diagonal process of multiplication, and might of course be dispensed with. By any process of multiplication we choose to adopt, the product of the trial factors $x^3 + mx^2 + nx + 1$ and $x^2 + px + 3$ will be found to be $x^5 + (m + p)x^4 + (n + mp + 3)x^3 + (3m + pn + 1)x^2 + (3n + p)x + 3$; and by equating the coefficients of the powers of x in this product with the coefficients of the corresponding powers in the given expression, we shall find the numerical values of m, n, and p as before.

ANSWERS TO APPENDIX.

INTRODUCTORY EXERCISES.

1. (10) $(2-ax)(2+ax)(4+a^2x^2)$
(14) $a^2(4b-1)(4b+1)$
(18) $2x^2(a-3y)(a+3y)$
(27) $a(x-y+z)(x+y-z)$
(29) $x(1-x)(1+x)$
2. (6) $x(x+2y)^2$; $x(x-2y)^2$
(11) $a(a-b)^2(a+b)^2$
(12) $3(x-a)^2(x+a)^2$
(13) $2x(x-2y)^2$
(14) $axy(a+xy)^2$
3. (10) $ax(x-a)(x^2+ax+a^2)$
(18) $(2+ax)(4-2ax+a^2x^2)$
(19) $2^3x^2(a-xy)(a^2+axy$
$+x^2y^2)$
(20) $x^2(x-1)(x^2+x+1)$

CHAPTER I.

1. (3) $(x^2-y)(xy-z)$
(5) $(x^2+2ay)(bx+y)(2x-y)$
(6) $(by+c)(ax+by-c)$
(7) $(z^2+a)(x-a)(x+a)$
(8) $(x-a)(x-b)(x+b)$
(x^2+ax+a^2)
3. (1) $\frac{1}{2}a,\ 2b$
(2) $a,\ b$
(3) $a+b,\ b+c$
(4) $a,\ -a,\ b^2$
4. (1) $(x+3a)(4x^2-9a^2)$
(x^2-4b^2)

CHAPTER II.

1. (4) $(5x-3)(2x+7)$; $(5x+3)$
$(2x-7)$
(10) $(a-x^2)(a-x)$; $(a+x^2)$
$(a+x)$
(11) $\begin{cases} xy-2a)(2x+3y) \\ (3x-y) \end{cases}$
$\{$(14) $2a-b+c)(a+2b-c)$
(15) $(xy-z)(xy+z)(x+y)$
$(2x+3y)$
(16) $(x-2y+3z)(x+2y-3z)$
2. (6) $a,\ \sqrt{a},\ -\sqrt{a}$
(7) $\frac{1}{2}b,\ -\frac{2}{3}b,\ \frac{1}{2}a,\ -\frac{1}{2}a$

CHAPTER III.

1. (5) $(x-3y)(x-2y)^2$
(8) $(x^2+7xy+2y^2)(x+2y)$
$(x+y+z)$
(9) $(a+x^2)(a+x)^2$
2. (1) $a,\ 2a,\ 3a$
(2) $-a,\ -2a,\ -3a$
(3) $a,\ \frac{1}{2}a,\ \frac{1}{3}a$

CHAPTER IV.

1. (3) $(x^2+2xy+5y^2)(x^2+7xy+y^2)$

(4) $(x^2-2xy+5y^2)(x^2-7xy+y^2)$

(10) $(5x^2-2xy+2y^2)(2x^2-9xy+7y^2)$

(11) $(x^2+ax+a^2)(x^2-ax+a^2)$

(18) $(3x^2+5xy-4y^2)(3x^2-5xy-4y^2)$

2. (3) $(2x-y)(x-2y)^2(3x-y)$

(5) $(a^2-ab+b^2)(a+b)^2$

(13) $\left.\begin{array}{l}(x^2+xy+y^2)\\ \quad(x^2-xy+y^2)(x-yz)\\ \qquad(x^2+xyz+y^2z^2)\end{array}\right\}$

3. (1) $5\pm\sqrt{22}$, $\frac{1}{2}(9\pm\sqrt{73})$

(2) $a(5\pm\sqrt{22})$, $\frac{a}{2}(9\pm\sqrt{73})$

(3) $3\pm\sqrt{2}$, $\frac{1}{2}(1\pm\sqrt{3}\sqrt{-1})$